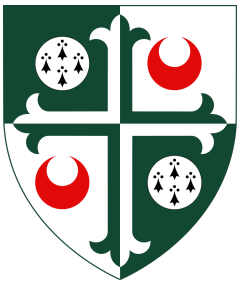
The Many Scales of Quantum Fields and Gravity



João F. Melo

Supervisor: Prof. Jorge E. Santos

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

Girton College

April 2023

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and the following.

Chapter 3 is based on [1] which was written in collaboration with Jorge E. Santos. The initial idea is not mine. I did all of the calculations and wrote the entirety of the document. The choices in direction during the production of the work were discussed together and the text was reviewed by both authors.

Chapter 4 is based on [2] which was written in collaboration with Jorge E. Santos. The initial idea is not mine. All calculations and code were done independently in parallel and using different methods by both authors to validate the final outcomes. The text was written in collaboration.

João F. Melo
April 2023

Acknowledgements

Despite the propaganda, a PhD is certainly not a thing one can do their own. The past four and a half years were long and complicated and I certainly would not be where I am without the support of several people.

Firstly, I would like to thank Jorge, my supervisor. Having many friends who are also PhD students I am well aware of how bad life can get if your supervisor isn't good. Not only did Jorge give very good academic support by being very wary of making sure I had enough projects to be able to finish on time (and not loose all motivation); and being very explicit about teaching how projects and ideas work, how to have healthy collaborations, and research ethics regarding authorship. But he also provided invaluable support through the pandemic and all my physical and mental health problems. It would have certainly been impossible to achieve this without that support.

There were also many Friends with a capital 'F', who were incredibly important throughout these years. Both in the pleasure and joy of their company, and their support when times were rough. I am of course talking about Bruno Bento (sem o Valeixo); the most powerful Júlia; Jorge and the many nights in London; Mariana (agora até temos foto!); and of course Maeve and Philip with the walks, the dinners, the movie nights and the Saturday morning calls of the Blue/Green/Red/RainbowSci Book Club. And I cannot forget Catarina who is both a young and old Friend and great company when exploring buildings which were totally not the moon, what are you talking about?

There were many other friends who were quite important in keeping my sanity (and theirs) through these crazy times. I would like to give a very warm apru to the European Union of Argyle Street, with a special thanks to the many late night discussions with Owain, which while often fruitless were nevertheless insightful. A real quick thanks to Cyprus Road. I would like to thank the best departmental badminton team in the history of pavilion B for the many fun sessions. I could also not forget to thank those who dungeoned some dragons and annihilated many tombs in the past and present (and hopefully future). All the members of the [Pun] Normal Chats for totally secret birthday parties, with a special mention to the geocaching with Roxana, and the traditional Aromi lunches with Miren (and, you know, sharing an office and being a good friend and all).

Também para a malta lá de casa. Às sessões de Catan, ou Geoguessr, ou só para conversar. Aos membros da banda PasseVit (ainda não acredito que ensaiámos durante o PhD). Aos muitos jantares de natal e chamadas com o pessoal das Cenas Fantásticas. E (embora estejam todos incluídos em menções anteriores) um agradecimento especial pelo (Meta)Física que me deu muito prazer não só a escrever, mas também a ler.

A final thanks to those who don't quite fit into any of the above categories, which certainly does not diminish their importance. Our semi-weekly calls are still very much a highlight of the day Sofia. And of course Leonhard, for the pleasure of sharing an office and the many interesting conversations we have had over the years.

Por último, como sempre, um agradecimento à família e em particular à minha mãe. Pelo apoio incondicional que me dão. Pela comida boa quando vou a casa. Pelas chamadas semanais que nos mantêm em contacto apesar da distância, e as chamadas mais frequentes com conversas sempre importantes e interessantes. Um grande abraço a todos.

Finally I would like to thank the Cambridge Trust for the Vice Chancellor's award which supported my first three years of study, and FCT for supporting the final year.

Abstract

It is a basic fact that Nature behaves differently at different scales. What happens at the human scale looks rather different to the motion of planets or the dynamics of subatomic particles. Rather remarkably, understanding this seemingly elementary phenomenon led to one of the most profound paradigm shifts in Physics and to the most accurate agreement between theory and experiment in Science. All thanks to the powerful tool of the Renormalisation Group.

However, in the presence of gravity this becomes rather challenging as scales become outputs of the dynamics rather than inputs. This thesis covers several studies regarding this intersection, trying to understand the interplay between scales, quantum fields, and gravity.

We begin with an overview of renormalisation techniques to set the stage for the rest of the discussion. We then delve into the AdS/CFT correspondence, first studying a local version of the renormalisation group and testing the quantum renormalisation group proposal. After that we analyse the surprising implications conformal and supersymmetry on the field theory side have on string theoretic corrections for its gravitational dual. Finally we explore the world of finite temperature quantum field theory and the subtleties behind correctly setting initial conditions.

Table of contents

List of figures

List of tables

Chapter 1

Introduction I: Quantum Gravity debunked debunked

In popular science quantum gravity is held as the ultimate theoretical physics question. As having this fundamental inconsistency that no one knows what to do about. But this idea is not too hard to debunk, quantum gravity is either already solved or not very interesting.

There isn't any fundamental inconsistency. It just so happens that gravitational interactions are irrelevant and therefore signal the theory breaks down at a certain energy scale: the Planck scale. So for anything below the Planck scale we already know how to do quantum gravity: we just use effective field theory techniques [4]. And the Planck scale is so far beyond our experimental reach that it seems like it is not very interesting to ponder about what might happen then. We have much more to worry us in the meantime.

But allow me to debunk the debunk. Not only is the above reasoning a bit too quick, but also there are other contexts when both quantum and gravitational effects are important, and we certainly do not have a good handle on those.

1.1 The difficulties with quantum fields in classical gravitational backgrounds

You just need to turn on a classical gravitational background to start to see cracks in our understanding. For free theories, we might think everything is settled, after all we have a perfectly rigorous understanding of those theories via algebraic QFT [5]. However, there is a very big difference between the rigorous and the practical.

The fundamental issue is well known: in the absence of an everywhere time-like Killing vector field there is an ambiguity in defining the notion of particle and therefore we lack

a preferred vacuum state [5]. We have known this long enough to have mostly gotten our heads around the conceptual hurdles, but the issue is also of a practical nature. Even for free theories, a calculation that would be almost trivial in flat spacetime can become extremely complicated when performed in a curved spacetime.

Let us consider, as an example, the calculation of the expectation value of the stress tensor. At first glance it seems fairly straightforward. Yes, there are a few conceptual difficulties but it's a subject of textbooks and fairly old textbooks at that [5, 6]. However, the techniques we understand and know how to use are extremely unwieldy.

Very briefly, the issue lies in renormalisation. We can calculate $\langle\phi(x_1)\phi(x_2)\rangle$ straightforwardly, but to calculate $\langle T_{\mu\nu}\rangle$ we need $\langle\phi^2(x)\rangle$, which diverges. In flat spacetime we can use normal ordering, but in curved spacetime this is more complicated. The standard way to proceed is to use an equivalence principle-type argument, and subtract the part of the divergences that behave the same way as divergences do in Minkowski spacetime. This is called Hadamard renormalisation [5, 6].

This is all well and good, but, to calculate the 2-point function with the correct vacuum we need to use a mode expansion (which in Minkowski spacetime would be a Fourier expansion). But the Hadamard expansion is written in spacetime variables and is completely independent of the mode expansion. To perform these calculations in practice we have to match these two and carefully subtract the necessary divergences. But subtracting two independent expansions is extremely labour intensive.

On the one hand, it seems like this is merely a technical issue, we have fewer symmetries so of course computations will be more involved. On the other, it is such a steep increase of difficulty that it may hint there are some conceptual subtleties regarding renormalisation in curved spacetimes that we have not fully addressed.

As you can imagine, if free theories are already this complicated adding interactions certainly won't help. We no longer have such a rigorous understanding of these theories, and in several contexts there are secular divergences which plague ordinary perturbative calculations. In recent years there has been a lot of progress in de Sitter spacetime [7–20] but, once again, all the solutions require extremely complicated and careful formulations of effective theories to rephrase the perturbative expansion. And this is for de Sitter, a maximally symmetric spacetime.

We can see that even before we make gravity dynamical we already have a plethora of technical and conceptual hurdles to clear. Quantum field theories in classical gravitational backgrounds are still quite poorly understood.

1.2 Some hints at issues with a gravitational EFT

The justification for using standard EFT techniques below the Planck scale is also under threat. Starting with the infamous information paradox, the crux of the matter is the combination of the 2nd law of black hole thermodynamics and Hawking's calculation for the temperature of a black hole [5, 6]. The conclusion of both of these facts is that back holes should be viewed as a thermodynamic object with entropy:

$$S_{\mathrm{BH}} = \frac{A}{4G_N} \tag{1.1}$$

where A is the black hole area and G_N is Newton's constant.

If we then believe there is a theory of quantum gravity then black holes should be described as ordinary quantum systems with S_{BH} degrees of freedom. This idea is sometimes called the 'central dogma'.

The issue arises when we try to consider black hole evaporation. For the sake of the argument let us imagine a black hole collapse from a pure state. After some time has passed we will have some degrees of freedom describing the black hole (which we do not have access from the outside) and some degrees of freedom describing the outgoing thermal radiation. Entanglement between these two regions is that gives us a non-trivial entropy. However, as the black hole radiates energy we anticipate it will gradually shrink, thereby reducing it's entropy, while the entropy of the radiation keeps increasing[1] [21].

The problem is that after some point in time, the entropy of the outgoing thermal radiation exceeds S_{BH}. Therefore we simply will not have enough degrees of freedom to entangle with to allow us to obtain a pure state for the whole system. And what is more, this point in time, called the Page time, can happen when the black hole is still a lot bigger than the Planck scale. At these scales we expect our EFT description to hold, and yet, we are finding inconsistencies.

This is the main idea behind the information paradox. And, despite recent progress, we still do not have a full solution. A review, including said recent progress and additional subtleties which we have not discussed, can be found in [21].

And this is not all that plagues a putative gravitational EFT. There has been a lot of recent work (with widely varying levels of rigour) on the 'swampland conjectures'. They are a set of non-trivial consistency conditions that quantum gravity theories should satisfy, which crucially cannot be derived from EFT arguments alone [22–24].

Just to give an example, let us consider the weak gravity conjecture. This one has a more solid basis of evidence, and it will also play a minor role in chapter 3. For gravity coupled to

[1] The entropy of the radiation has to increase because the overall entropy cannot decrease.

electromagnetism there is a mass to charge ratio such that the black hole has zero temperature and therefore it will not evaporate, these are called 'extremal black holes'. They are naively stable. However, if they were exactly stable and there was no way to radiate their charge away, then we would end up with a very strong degeneracy in charged states which violates certain entropy bounds [24]. The solution is simple, we need a particle in the spectrum who can radiate away the black hole charge. It is straightforward to show it then must satisfy:

$$\frac{Q}{M} \geq \left(\frac{Q}{M}\right)_{\text{Extremal BH}} \tag{1.2}$$

For this particle the gravitational force will be weaker than the electromagnetic, hence the name of the conjecture.

However, this then implies that an EFT of gravity weakly coupled to a $U(1)$-gauge theory will only be consistent if it also includes said particle. Furthermore, the smaller the gauge coupling, the smaller the charge and therefore the smaller the mass of said missing particle. So at these potentially quite low energies, far below the Planck mass, the EFT will not see anything wrong, but we know there must be an issue.

In chapter 3, we present an example of how higher derivative corrections can solve this. These corrections can shift the charge to mass ratio in such a way to make it possible for small black holes to serve the role of this missing particle.

This is merely a conjecture, supported by informal arguments and examples from string theory. Far beyond a rigorous proof or paradox. However, it still hints at interestingly non-trivial obstacles to writing a fully consistent gravitational EFT.

1.3 A lighting introduction to the AdS/CFT correspondence

And lastly in the list of examples of how gravity *with* quantum is interesting, even if quantum gravity is boring, we have the AdS/CFT correspondence [25–28]. The ideas behind this duality are intricate because they require a deep understanding of supergravity, conformal field theory and string theory. However, we can give an intuition behind the statement of the correspondence by just analysing anti-de Sitter (AdS) spacetime. Whenever we make use of (an example of) this duality in the following chapters we will always introduce the relevant models and the particular details of the correspondence that we will use.

Anti-de Sitter spacetime is the maximally symmetric spacetime with negative cosmological constant. In the Poincaré patch, the metric is given by:

$$ds^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right) \tag{1.3}$$

where L is a parameter with dimension of length related to the cosmological constant, and $\eta_{\mu\nu}$ is the ordinary Minkowski metric.

At $z = 0$ we have a conformal boundary. However, in opposition to the asymptotically flat spacetimes' conformal boundary, massless particles can travel to this boundary and return in finite proper time. In some sense, light can bounce off infinity and come back in finite time. Therefore, the initial value problem in anti-de Sitter spacetime has to also include suitable boundary conditions at $z = 0$. This is why this spacetime is sometimes called a 'perfectly covariant box'.

Crucially, the geometry of this boundary is only fixed up to conformal transformations, as the name indicates. Therefore, the boundary conditions we need to specify in order to solve the equations of motion in the bulk are invariant under conformal transformations. Also importantly, the boundary has Lorentzian signature and it is indeed a Lorentzian spacetime in its own right albeit in one fewer dimension.

The duality relates the gravitational action evaluated on a solution of the equations of motion with given boundary conditions to the partition function of a quantum field theory (which is by necessity conformally invariant) where those boundary conditions now play the role of sources[2]:

$$\left\langle \exp\left(\int \mathrm{d}^d x \, \mathscr{O} \phi^{(0)} \right) \right\rangle_{\mathrm{QFT}} = \mathrm{e}^{-S_{\mathrm{Gravity}}} \Big|_{\lim_{z\to 0} \bar{\phi}(z,x) z^{\Delta - d} = \phi^{(0)}(x)} . \tag{1.4}$$

In summary, in the LHS we have a conformal field theory with certain sources turned on and we just compute the normal expectation value. In the RHS we have a gravitational theory where we have fixed boundary conditions (related to the sources, and spacetime metric, from the LHS), we then solve the equations of motion and evaluate the action on this solution. Justification for this equation comes from string theory, but it is still conjectural, no formal proof is known, there is, however, an ample body of evidence in favour of it. In most examples both sides of this duality have a certain degree of supersymmetry. A pedagogical summary of the construction and applications can be found in [29].

This correspondence is as puzzling as it is useful because it relates strong to weak coupling. So, on the one hand, it is extremely hard to provide a rigorous proof because we have a very feeble grasp on strongly coupled theories. On the other hand it is extremely useful because it can be used as a tool to examine the very poorly understood strongly coupled gauge theories or even turn it around and use to study quantum gravity.

[2] In this equation Δ is the conformal dimension of the field and the factor of $z^{\Delta - d}$ is needed to fix the boundary conditions appropriately

1.4 Summary of the thesis

In this tangle of confusion and complications how should we proceed? In all earnestness, I don't know. The correct approach is probably to try everything at the same time and seeing what sticks. However, time is limited and so we must choose an approach. In this thesis I will take lessons from what worked in the past for ordinary quantum field theory and focus on renormalisation techniques. Symmetries will still make an appearance but by in large the focus will be on the interplay of different scales.

Given this emphasis on renormalisation, the second half of the introduction will go over the basics of renormalisation in flat spacetime. The material is based on my lecture notes on the topic [30]. It is not original research but the narrative and specific arguments may be new and hopefully are enlightening even for experts in the field. The matter is somewhat introductory but it is also useful to understand very deeply a simpler example to have that in mind when studying the more complicated ones.

Then we move on to AdS/CFT and in particular how RG flow in the CFT side appears to correspond to radial evolution in the AdS side. We will use the BFSS theory to test a particular proposal to make this more precise: quantum RG. And eventually rule it out as a possibility. In the meantime we develop a local formulation of RG which is useful in its own right. This chapter is based on work written in collaboration with Jorge E. Santos [1].

After the null result we stop, analyse, and test a concrete prediction from this duality. Namely, the way the black hole entropy was calculated in the CFT side implies it is independent of the strength of the coupling (this fact is what made those calculations possible). However, this then implies that the black hole entropy is independent of stringy corrections in the AdS side. In this chapter we confirm this prediction. This chapter is based on work written in collaboration with Jorge E. Santos [2].

Finally we turn our attention to QFT in curved spacetime and try to tackle secular divergences. Or we would have done so if we hadn't come across several misconceptions in the finite temperature literature about how to properly include interactions in the initial conditions (crucial for the secular growth calculations). We find that, despite the prevailing assumptions, we cannot neglect interactions when setting up the initial finite temperature state, even in the infinite past. This chapter is based on my single-author paper [31].

Chapter 2

Introduction II: A primer on renormalisation

In the study of physics there are two universal concepts that are absolutely crucial for us to really understand what is going on: symmetry and scale. Both deserve quite a lot of attention and care but this thesis will mostly focus on that of scale.

It is then instructive for us to consider the simplest case in great detail. Only in this way can we try to tease out what are merely technicalities and what are the really important points. We shall delve into the study of time-ordered correlators of a single scalar field in Minkowski spacetime[1]. The topics may seem somewhat introductory but all of the points made here will play an explicit role in the chapters to follow.

This chapter is based on my lecture notes on the topic [30] with some updated discussions and perspectives. It is mostly not original research but merely a retelling of what is common knowledge among experts. I will endeavour to be extremely explicit about all the intricacies and subtleties in this game and how all the different scales interact. Always using the simplest possible examples and ideas to move forward.

2.1 The need for a regulator

We are interested in the following Hamiltonian:

$$H = \int \mathrm{d}^d x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi) \right) \tag{2.1}$$

[1] We do this with the LSZ reduction formula in mind which tells us how to relate time-ordered expectation values with S-matrix elements.

where Π is the momentum conjugate to ϕ. For simplicity we shall consider non-derivative interactions, including derivatives takes a bit more work to convert from Hamiltonian to Langrangian perspectives, but the main physical picture is not affected.

We will be interested in calculating, time-ordered expectation values:

$$G^{(n)}(x_1,\ldots,x_2) = \langle\Omega|\,\mathscr{T}(\phi(x_1)\ldots\phi(x_n))\,|\Omega\rangle \tag{2.2}$$

where $|\Omega\rangle$ is the vacuum state and $\mathscr{T}$ denotes time-ordering.

To simplify matters we will Fourier transform with respect to the spatial variables, which effectively reduces our system to the 1-dim case. To convert back to D-dim we just need to make $m \to E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}$ and integrate[2] over $\boldsymbol{p}$. We therefore write:

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi) \tag{2.3}$$

Now we are ready to convert (2.2) to a path integral. Without loss of generality let us assume $t_1 < t_2 < \cdots < t_n$. Writing the time evolution operators explicitly and using the fact the Hamiltonian is time-independent we get

$$G^{(n)}(t_1,\ldots,t_2) = \langle\Omega|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_n)}\phi\mathrm{e}^{-\mathrm{i}H(t_n-t_{n-1})}\cdots\mathrm{e}^{-\mathrm{i}H(t_2-t_1)}\phi\mathrm{e}^{-\mathrm{i}H(t_1-t_0)}\,|\Omega\rangle \tag{2.4}$$

We have introduced some spurious $t_0 < t_1$ and $t_f > t_n$. These do not affect the final answer given that, even in the Schrodinger picture, the vacuum state, being an eigenstate of the Hamiltonian, is independent of time. These are merely book-keeping parameters and should drop out of the final calculation.

Inserting $n+2$ copies of the identity $\int \mathrm{d}\phi\,|\phi\rangle\langle\phi|$ we find

$$\begin{aligned} &G^{(n)}(t_1,\ldots,t_2) = \\ &= \int\left(\prod_{j=0}^{n+1}\mathrm{d}\phi_j\right)\langle\Omega|\phi_{n+1}\rangle\,\langle\phi_{n+1}|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_n)}\,|\phi_n\rangle\,\phi_n\,\langle\phi_n|\,\mathrm{e}^{-\mathrm{i}H(t_n-t_{n-1})}\,|\phi_{n-1}\rangle\cdots \\ &\cdots\phi_1\,\langle\phi_1|\,\mathrm{e}^{-\mathrm{i}H(t_1-t_0)}\,|\phi_0\rangle\,\langle\phi_0|\Omega\rangle \end{aligned} \tag{2.5}$$

The basic unit in this expression is $\left\langle\phi_j\right|\mathrm{e}^{-\mathrm{i}H(t_j-t_{j-1})}\left|\phi_{j-1}\right\rangle$ so let us calculate that. The trick is to divide the interval $[t_{j-1},t_j]$ in N slices of size $\Delta t = \frac{t_j-t_{j-1}}{N}$ and at the end take the

[2]We are intentionally being a bit cavalier. This reduction is only straightforward if it was an ordinary sum. This continuous integration is precisely the origin of most of the difficulties with field theories.

limit $N \to \infty$. We then have

$$\begin{aligned}
&\left\langle \phi_j \right| \mathrm{e}^{-\mathrm{i}H(t_j - t_{j-1})} \left| \phi_{j-1} \right\rangle = \left\langle \phi_j \right| \left(\mathrm{e}^{-\mathrm{i}H\Delta t} \right)^N \left| \phi_{j-1} \right\rangle = \\
&= \int \left(\prod_{k=1}^{N-1} \mathrm{d}\phi_{(k)} \right) \left\langle \phi_j \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(N-1)} \right\rangle \left\langle \phi_{(N-1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(N-2)} \right\rangle \cdots \left\langle \phi_{(1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{j-1} \right\rangle
\end{aligned} \tag{2.6}$$

where, in going to the last line, we once more inserted the identity between each Hamiltonian. It seems like we started, however, now we can take advantage of the limit $N \to \infty$ and approximate Δt to be small. To leading order in Δt the Baker-Campbell-Hausdorff (BCH) formula gives

$$\mathrm{e}^{-\mathrm{i}H\Delta t} \approx \exp\left(-\mathrm{i}\Delta t \frac{1}{2} \Pi^2 \right) \exp\left(-\mathrm{i}\Delta t \frac{1}{2} m^2 \phi^2 + V_{int}(\phi) \right) \tag{2.7}$$

The second trick is inserting the identity in the form $\int \frac{\mathrm{d}\Pi}{2\pi} |\Pi\rangle\langle\Pi|$, which gives

$$\begin{aligned}
\left\langle \phi_{(k+1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(k)} \right\rangle &= \int \frac{\mathrm{d}\Pi_{(k)}}{2\pi} \left\langle \phi_{(k+1)} \middle| \Pi_{(k)} \right\rangle \left\langle \Pi_{(k)} \right| \exp(-\mathrm{i}H\Delta t) \left| \phi_{(k)} \right\rangle \approx \\
&\approx \int \frac{\mathrm{d}\Pi_{(k)}}{2\pi} \left\langle \phi_{(k+1)} \middle| \Pi_{(k)} \right\rangle \left\langle \Pi_{(k)} \middle| \phi_{(k)} \right\rangle \cdot \\
&\quad \cdot \exp\left(-\mathrm{i}\Delta t \left(\frac{1}{2} \Pi_{(k)}^2 + \frac{1}{2} m^2 \phi_{(k)}^2 + V_{int}\left(\phi_{(k)}\right) \right) \right)
\end{aligned} \tag{2.8}$$

where we went from operators to eigenvalues by acting with $\exp\left(-\mathrm{i}\Delta t \left(\frac{1}{2} m^2 \phi^2 + V_{int}(\phi)\right)\right)$ on the right and $\exp\left(-\mathrm{i}\Delta t \frac{1}{2} \Pi^2\right)$ on the left.

Using $\langle \Pi | \phi \rangle = \exp(-\mathrm{i}\Pi\phi)$ we get

$$\begin{aligned}
\left\langle \phi_{(k+1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(k)} \right\rangle &\approx \int \frac{\mathrm{d}\Pi_{(k)}}{2\pi} \exp\left(\mathrm{i}\Pi_{(k)} \left(\phi_{(k+1)} - \phi_{(k)} \right) \right) \cdot \\
&\quad \cdot \exp\left(-\mathrm{i}\Delta t \left(\frac{1}{2} \Pi_{(k)}^2 + \frac{1}{2} m^2 \phi_{(k)}^2 + V_{int}\left(\phi_{(k)}\right) \right) \right)
\end{aligned} \tag{2.9}$$

Computing this integral is fairly straightforward. First we complete the square by defining

$$x = \Pi_{(k)} - \frac{\phi_{(k+1)} - \phi_{(k)}}{\Delta t} \tag{2.10}$$

to get

$$\left\langle \phi_{(k+1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(k)} \right\rangle \approx \exp\left(-\mathrm{i}\Delta t \left(-\frac{(\phi_{(k+1)} - \phi_{(k)})^2}{2\Delta t^2} + \frac{1}{2} m^2 \phi_{(k)}^2 + V_{int}\left(\phi_{(k)}\right) \right) \right) \cdot \cdot \int \frac{\mathrm{d}x}{2\pi} \exp\left(-\mathrm{i}\Delta t \frac{1}{2} x^2 \right) \tag{2.11}$$

This would be an ordinary Gaussian integral, however, the coefficient is pure imaginary so it is worth revising how to compute (the Cauchy principal value of) this integral. We write a contour in the complex place $C = C_1 + C_2 + C_3$ such that

$$C_1 : \; z = x, \; x \in [0, R] \tag{2.12}$$

$$C_2 : \; z = R\mathrm{e}^{\mathrm{i}\theta}, \; \theta \in [0, \theta_2] \tag{2.13}$$

$$C_3 : \; z = r\mathrm{e}^{\mathrm{i}\theta_2}, \; r \in [R, 0] \tag{2.14}$$

for $R > 0$, $\theta_2 < 0$, also note that $\Delta t > 0$.

Due to the absence of poles we have:

$$0 = \int_C \mathrm{d}z \exp\left(-\mathrm{i}\frac{\Delta t}{2} z^2 \right) = \tag{2.15}$$

$$= \int_0^R \mathrm{d}x \exp\left(-\mathrm{i}\frac{\Delta t}{2} x^2 \right) + \int_0^{\theta_2} \mathrm{d}\theta \, \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta} \exp\left(-\mathrm{i}\frac{\Delta t}{2} R^2 \mathrm{e}^{2\mathrm{i}\theta} \right) + + \int_R^0 \mathrm{d}r \, \mathrm{e}^{\mathrm{i}\theta_2} \exp\left(-\mathrm{i}\frac{\Delta t}{2} r^2 \mathrm{e}^{2\mathrm{i}\theta_2} \right) \tag{2.16}$$

The trick is now to choose θ_2 such that the integral along C_2 vanishes as $R \to \infty$ and the integral along C_3 is an ordinary Gaussian integral with a negative coefficient. It is not too hard to see that $\theta_2 = -\frac{\pi}{4}$ is the correct choice. We therefore conclude:

$$\int_0^\infty \mathrm{d}x \exp\left(-\mathrm{i}\frac{\Delta t}{2} x^2 \right) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \int_0^\infty \mathrm{d}r \exp\left(-\frac{\Delta t}{2} r^2 \right) = \sqrt{\frac{\pi}{2\mathrm{i}\Delta t}} \tag{2.17}$$

and therefore:

$$\left\langle \phi_{(k+1)} \right| \mathrm{e}^{-\mathrm{i}H\Delta t} \left| \phi_{(k)} \right\rangle \approx \frac{\exp\left(-\mathrm{i}\Delta t \left(-\frac{(\phi_{(k+1)} - \phi_{(k)})^2}{2\Delta t^2} + \frac{1}{2} m^2 \phi_{(k)}^2 + V_{int}\left(\phi_{(k)}\right) \right) \right)}{\sqrt{2\pi\mathrm{i}\Delta t}} \tag{2.18}$$

The main point to make here is that this integral is completely finite. There was no need to Wick rotate to Euclidean time[3].

Putting all of these pieces together we find

$$\begin{aligned}&\left\langle \phi_j \right| \mathrm{e}^{-\mathrm{i}H(t_j - t_{j-1})} \left| \phi_{j-1} \right\rangle = \\ &= \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \mathrm{d}\phi_{(k)} \, \frac{\exp\left(-\mathrm{i}\Delta t \left(-\frac{(\phi_{(k+1)} - \phi_{(k)})^2}{2\Delta t^2} + \frac{1}{2} m^2 \phi_{(k)}^2 + V_{int}\left(\phi_{(k)}\right) \right) \right)}{\sqrt{2\pi \mathrm{i}\Delta t}}\end{aligned} \tag{2.19}$$

with the understanding that $\phi_{(0)} = \phi_{j-1}$ and $\phi_{(N)} = \phi_j$.

We now take the limit inside of the integral and interpret this as a functional integral

$$\left\langle \phi_j \right| \mathrm{e}^{-\mathrm{i}H(t_j - t_{j-1})} \left| \phi_{j-1} \right\rangle = \int_{\phi(t_{t-1})=\phi_{j-1}}^{\phi(t_j)=\phi_j} \mathscr{D}\phi \; \mathrm{e}^{\mathrm{i}S[\phi]} \tag{2.20}$$

where

$$S[\phi] = \int_{t_{j-1}}^{t_j} \mathrm{d}t \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 - V_{\mathrm{int}}(\phi) \right) \tag{2.21}$$

is the action. You may be rightly worried regarding the validity of this limiting procedure. If there are a finite number of spatial degrees of freedom (that is, if we are truly in a 1-dim case), then it can be shown that this works using a Wiener measure [32], or, alternatively, by considering non-differentiable field configurations. Nevertheless, this is not necessarily true for higher dimensions, which is indeed the source of the issues this chapter will tackle.

We still need to plug this into (2.5). To do this we just need to note a few things. Firstly, in the functional integration, the fields now depend on time, the boundary conditions stipulate that $\phi(t_j) = \phi_j$. Therefore the factors of ϕ_j can be replaced by $\phi(t_j)$. Secondly, integrating over field configurations between t_{j-1} and t_j, then integrating over field configurations between t_j and t_{j+1} while matching the field values at t_j, and finally integrating over all possibilities at this matching time, is the same as integrating over all field configurations between t_{j-1} and t_{j+1}. Therefore plugging (2.20) into (2.5) gives us

$$G^{(n)}(t_1, \ldots, t_2) = \int \mathrm{d}\phi_0 \, \mathrm{d}\phi_f \left\langle \Omega \right| \left. \phi_f \right\rangle \left\langle \phi_0 | \Omega \right\rangle \int_{\phi(t_0)=\phi_0}^{\phi(t_f)=\phi_f} \mathscr{D}\phi \; \mathrm{e}^{\mathrm{i}S[\phi]} \prod_{j=1}^{n} \phi(t_j) \tag{2.22}$$

If we pick some arbitrary t_0 and t_f it is quite hard to calculate the position representation of the vacuum state including interactions. However, we can use our freedom to choose them and take the limits $t_0 \to -\infty$ and $t_f \to \infty$ to simplify things. We can then write, expanding in

[3] This fact is important for the calculations in chapter 5

the eigenvalues of the Hamiltonian $|E_n\rangle$

$$\langle\phi_f|\,\mathrm{e}^{-\mathrm{i}Ht_f} = \sum \langle\phi_f|E_n\rangle\,\langle E_n|\,\mathrm{e}^{-\mathrm{i}E_n t_f} \tag{2.23}$$

If we multiply the Hamiltonian by $(1-\mathrm{i}\varepsilon)$, then in the limit $t_f \to \infty$ all contributions except for the vacuum (the lowest energy state) are suppressed. If we therefore multiply the Hamiltonian by $(1-\mathrm{i}\varepsilon)$ and take the limit $\varepsilon \to 0^+$, regardless of the field boundary conditions at t_0 or t_f we are computing vacuum expectation values.

It is straightforward to check that repeating the above reasoning using instead the Hamiltonian $H_\varepsilon = (1-\mathrm{i}\varepsilon)H$ we get the same path integral but with the action

$$S_\varepsilon[\phi] = \int \mathrm{d}t \left(\frac{1}{2}\frac{\dot\phi^2}{1-\mathrm{i}\varepsilon} - \frac{1}{2}(1-\mathrm{i}\varepsilon)m^2\phi^2 - (1-\mathrm{i}\varepsilon)V_{\mathrm{int}}(\phi)\right) \tag{2.24}$$

We're almost there. All we have left to do is compute the path integral. Luckily for us, we know of Feynman diagrams. We begin by introducing sources to our action to write:

$$G^{(n)}(t_1,\ldots,t_2) = \int \mathscr{D}\phi\ \mathrm{e}^{\mathrm{i}S_\varepsilon[\phi]} \prod_{j=1}^{n}\phi(t_j) = \prod_{j=1}^{n}\left(-\mathrm{i}\frac{\delta}{\delta J(t_j)}\right)\int \mathscr{D}\phi\ \mathrm{e}^{\mathrm{i}S_J[\phi]} \tag{2.25}$$

where

$$S_J[\phi] = S_\varepsilon[\phi] + \int \mathrm{d}t\, J\phi \tag{2.26}$$

Therefore we have reduced our job to that of calculating the generating functional:

$$Z[m, V_{\mathrm{int}}, J] = \int \mathscr{D}\phi\ \mathrm{e}^{\mathrm{i}S_J[\phi]} \tag{2.27}$$

Employing a similar trick we get

$$Z[m, V_{\mathrm{int}}, J] = \exp\left(-\mathrm{i}\int \mathrm{d}t\,(1-\mathrm{i}\varepsilon)V_{\mathrm{int}}\left(\frac{\delta}{\delta J}\right)\right)\int \mathscr{D}\phi\ \mathrm{e}^{\mathrm{i}S_{0,J}[\phi]} \tag{2.28}$$

where

$$S_{0,J}[\phi] = \int \mathrm{d}t \left(\frac{1}{2}\frac{\dot\phi^2}{1-\mathrm{i}\varepsilon} - \frac{1}{2}(1-\mathrm{i}\varepsilon)m^2\phi^2 + J\phi\right) \tag{2.29}$$

Now the only functional integral we need to compute is quadratic and that is easy enough. First we complete the square by writing

$$\Phi(t) = \phi(t) + \int \mathrm{d}t'\, G(t,t')J(t') \tag{2.30}$$

for G obeying

$$\left(-\frac{1}{1-\mathrm{i}\varepsilon}\partial_t^2-(1-\mathrm{i}\varepsilon)m^2\right)G(t,t')=\delta(t-t') \tag{2.31}$$

with more or less arbitrary boundary conditions. This equation is very easy to solve using Fourier transforms:

$$\tilde{G}(\omega)=\frac{1-\mathrm{i}\varepsilon}{\omega^2-(1-\mathrm{i}\varepsilon)^2m^2} \tag{2.32}$$

And now we see the role of the ε more clearly. It serves to shift the would be pole at $\omega^2=m^2$ away from the real axis. This way, in the limit $\varepsilon\to 0^+$ we know what is the correct contour to choose around the poles when integrating over momenta. The specific expression we found agrees with a more careful treatment using Lindblad evolution [33], but, in the limit, the actual details of the ε dependence won't matter, just how to go around the poles. We will then follow what is common in textbooks and write more simply

$$\tilde{G}(\omega)=\frac{1}{\omega^2-m^2+\mathrm{i}\varepsilon} \tag{2.33}$$

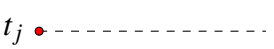
the factor of $(1-\mathrm{i}\varepsilon)$ in the potential is completely unimportant so we will drop it.

Assuming the boundary conditions are such that we can integrate by parts freely and up to normalisation we get

$$Z[m,V_{\mathrm{int}},J]=\exp\left(-\mathrm{i}\int \mathrm{d}t\, V_{\mathrm{int}}\left(\frac{\delta}{\delta J(t)}\right)\right)\exp\left(-\mathrm{i}\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}t'\,J(t)G(t,t')J(t')\right) \tag{2.34}$$

From here it is a matter of expanding both exponentials and doing dull combinatorics involving the orbit stabilizer theorem [30, 32]. The trick turns out to be to associate each term with a diagram, then our job is to draw all possible diagrams and add them together with appropriate symmetry factors which take into account the combinatorics. The rules are as follows:

1. For every field inside the correlation function you associate an external point[4].



2. Draw every possible diagram (up to the order required) connecting the external points.

3. Associate a number to each diagram using the following rules:

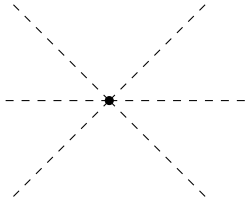
[4]Throughout the manuscript we shall use the convention that scalar fields are denoted with dashes and fermions with continuous lines. Gauge fields would be denoted with wavy lines.

- A line joining two points, called a *propagator*, carries a factor of

$$\mathrm{i}G(t,t') = \quad t \quad t'$$

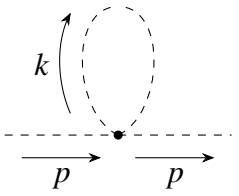
- You join several internal lines in a *vertex*. A vertex with k lines carries a factor of

$$-\mathrm{i}\partial_\phi^k V\Big|_{\phi=0} =$$



- Integrate over internal time/momenta
- Divide by the symmetry factor, which is the number of actions you can perform on a labelled diagram, that is, one where every end of the each propagator is labelled uniquely, which leave it completely unchanged (including the labelling). The actions are flipping a propagator's ends or exchanging it with another propagator.

Now we are ready to do calculations. Let us do the simplest non-trivial calculation in D dimensions. The first order correction to the 2-point function for a quartic interaction. In Fourier space, this looks like[5]:



$$= \frac{\lambda}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{k^2+m^2-\mathrm{i}\varepsilon} \tag{2.35}$$

The integrand grows as k^{D-3} therefore it diverges for $D > 2$. This is the simplest of the famous infinities that seemingly plague quantum field theory. They are generic for loop calculations and indicate that a Wiener measure like it worked for quantum mechanics will not save us. To be able to definite our quantum theory we need to tame these infinities.

[5] In Fourier space we usually omit the external propagators. This is because they are merely a multiplicative factor which doesn't affect the conclusions. This practice is commonplace because that is how you compute S-matrix elements.

2.2 The Wilsonian Renormalisation Group

The first option that comes to mind to get rid of these infinities is to introduce a regulator: some modification of the integral that would render it finite, in the hopes this would enable a better understanding of these contributions.

The choice of regulator (also called a renormalisation scheme) is somewhat arbitrary and arguably unphysical. We should be able to get out the same physics regardless of our choice. However, akin to coordinate systems or gauges our choices will certainly impact the difficulty of the calculations.

It seems at first sight as if we would only need to cut off or dampen high spatial momenta to avoid trouble. However, this breaks Lorentz invariance which may not be desirable. We will instead use the analytic properties of time-ordered correlators to Wick rotate to Euclidean space by defining $\tau = \mathrm{i}t$ and then introduce a hard cut-off for all momenta. In this way we are treating space and time equally and therefore preserving more symmetries[6].

In summary, we are considering the action

$$S[\phi] = \int \mathrm{d}^D x \left(\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right) \tag{2.36}$$

where the fields have finite support in momentum space:

$$\tilde{\phi}(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > \Lambda_0 \\ \tilde{\phi}(\boldsymbol{p}) & |\boldsymbol{p}| < \Lambda_0 \end{cases} \tag{2.37}$$

In this theory, momenta above the cut-off Λ_0 simply do not exist. If we need to ask questions with that high a momenta then our theory simply cannot answer them. Further, given the arbitrariness of our choice of regulator we might even be reasonably sceptical that our theory would be capable of describing momenta right up to the cut-off, since, if we get too close we might start getting artefacts from the precise way in which we regulated the theory. Therefore, external momenta in correlators should obey $|\boldsymbol{p}| \ll \Lambda_0$.

In this case, it really shouldn't make a difference whether we use the cut-off Λ_0 or whether we use a cutoff Λ which is slightly smaller, as long as $|\boldsymbol{p}| \ll \Lambda$ still holds. Let us build that theory.

We will split our field into two components, the useless/high energy/UV modes, ϕ^+, that have support between Λ and Λ_0 i.e.

[6] In other settings this still does not preserve all the symmetries one would wish to preserve and other regulators such as dimensional regularisation become more convenient.

$$\tilde{\phi}^+(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > \Lambda_0 \\ \tilde{\phi}(\boldsymbol{p}) & \Lambda < |\boldsymbol{p}| < \Lambda_0 \\ 0 & |\boldsymbol{p}| < \Lambda \end{cases} \tag{2.38}$$

and the useful/low energy/IR modes, ϕ^-, that have support below Λ,

$$\tilde{\phi}^-(\boldsymbol{p}) = \begin{cases} 0 & |\boldsymbol{p}| > \Lambda \\ \tilde{\phi}(\boldsymbol{p}) & |\boldsymbol{p}| < \Lambda \end{cases} \tag{2.39}$$

We also split the sources in the same way to get $J^\pm$ which source $\phi^\pm$ respectively. To codify the statement that ϕ^+ cannot appear in the external correlators we set J^+ to 0. The generating functional then looks like:

$$Z = \int \mathscr{D}\phi^- \mathscr{D}\phi^+ \, \mathrm{e}^{-S_{\Lambda_0}[\phi^+ + \phi^-] - \int J^- \phi^-} \stackrel{!}{=} \int \mathscr{D}\phi^- \, \mathrm{e}^{-S_\Lambda[\phi^-] - \int J^- \phi^-} \tag{2.40}$$

where in the last line we expressed our wish of writing the generating functional solely as an integration over the low energy modes. To do this we define the *Wilsonian effective action* as,

$$\mathrm{e}^{-S_\Lambda[\phi^-]} = \int \mathscr{D}\phi^+ \mathrm{e}^{-S_{\Lambda_0}[\phi^+ + \phi^-]} \tag{2.41}$$

Note that the $\int J^- \phi^-$ term in the exponential is completely unaffected by this integral.

This path integral looks very scary, but we can make some progress. In momentum space (2.36) looks like:

$$S[\tilde{\phi}] = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{2} \tilde{\phi}(-\boldsymbol{p}) \left(\boldsymbol{p}^2 + m^2\right) \tilde{\phi}(\boldsymbol{p}) + \frac{\lambda}{4!} \int \prod_{i=1}^{4} \left(\frac{\mathrm{d}^D p_i}{(2\pi)^D} \tilde{\phi}(\boldsymbol{p}_i) \right) \delta^{(D)} \left(\sum_{i=1}^{4} \boldsymbol{p}_i \right) \tag{2.42}$$

Note how, in the quadratic term both fields have the same momentum. Therefore, given ϕ^+ and ϕ^- have non-intersecting support in momentum space we conclude there cannot be any quadratic mixing between the two fields, that is:

$$S_{\Lambda_0}[\phi^+ + \phi^-] = S_{\text{free}}[\phi^+] + S_{\text{free}}[\phi^-] + S_{\text{int}}[\phi^+, \phi^-] \tag{2.43}$$

where

$$S_{\text{free}}[\phi] = \int \mathrm{d}^D x \frac{1}{2} \left((\boldsymbol{\nabla}\phi)^2 + m_0^2 \phi^2 \right) \tag{2.44}$$

$$S_{\text{int}}[\phi^+,\phi^-] = \int \mathrm{d}^D x \frac{\lambda_0}{4!}(\phi^+ + \phi^-)^4 \tag{2.45}$$

Now, since $S_{\text{free}}[\phi^-]$ is independent of ϕ^+, we can pull it out of the integral. Defining $\Delta S_\Lambda[\phi^-] = S_\Lambda[\phi^-] - S_{\text{free}}[\phi^-]$, we get

$$\mathrm{e}^{-\Delta S_\Lambda[\phi^-]} = \int \mathscr{D}\phi^+ \mathrm{e}^{-S_{\text{free}}[\phi^+]-S_{\text{int}}[\phi^+,\phi^-]} \tag{2.46}$$

which has the same form as the generating functional for connected correlators [32]. We then conclude that, at least in perturbation theory, we can obtain ΔS_Λ by evaluating connected Feynman diagrams with ϕ^- playing the role of external sources.

The outcome of these calculations will then generically look like a series expansion in powers of ϕ^-. The connectedness of the diagrams will mean each term will get a Dirac-δ enforcing momentum conservation. However, we might still get a free function of the momenta involved. Nevertheless, we can also expand these functions in powers of the momenta[7]. In position space, the overall Dirac-δ will mean we can express everything in the usual way with an single spacial integration and local interaction terms; the powers of momenta will turn into derivatives of the spacial fields. All in all we get something that schematically looks like this:

$$S_\Lambda[\phi^-] = \int \mathrm{d}^D x \left(\frac{1}{2} Z'_\phi (\boldsymbol{\nabla}\phi^-)^2 + \frac{1}{2} m'^2 (\phi^-)^2 + \frac{\lambda'}{4!}(\phi^-)^4 + \dots \right) \tag{2.47}$$

This form justifies calling this an action. It really is just an ordinary action (albeit with potentially infinitely many terms) for the field ϕ^-. All the couplings in this action are different than the original ones, as we change the cut-off we change the couplings, flowing in the space of all allowed theories.

At first sight this seems quite concerning. Our couplings are completely dependent of this ad-hoc parameter we have introduced. But actually, it is this very dependence that makes everything consistent. In the end, we more or less just changed the order of integration. We could have kept all of the fields and integrated all in one go, instead we identified some useless UV fields and integrated over those first. The action may have changed, but the end result can't have. In terms of the generating functional we might want to write down something like:

$$Z(\Lambda_0, g_{0,a}) \stackrel{!}{=} Z(\Lambda, g'_a) \tag{2.48}$$

where $g_{0,a}$ and g'_a schematically represent all the couplings and parameters in the two theories.

[7] Or more precisely in powers of $\boldsymbol{p}/\Lambda$ which we have assumed to be small.

But we have to be careful when writing this. By changing the cut-off we are interfering with both the spacetime integration inside the action and the functional integration. In (2.48) as written we not comparing the same function with two different inputs. The two sides of the equation actually have two different functions because internally the two integrals are different. We are comparing apples and oranges. If we want to write something on the lines of (2.48) we must fix the normalisation of both integrals.

To fix the spacetime integral we rescale our spacetime variables to ensure the cut-off is the same:

$$\boldsymbol{x} \to \boldsymbol{x}' = \frac{\Lambda}{\Lambda_0}\boldsymbol{x}, \quad \boldsymbol{p} \to \boldsymbol{p}' = \frac{\Lambda_0}{\Lambda}\boldsymbol{p} \tag{2.49}$$

which ensures that $|\boldsymbol{p}| = \Lambda \Rightarrow |\boldsymbol{p}'| = \Lambda_0$.

To fix the functional integration we have to fix the overall normalisation of our fields. This step is quite different depending on the particular theory and context at hand. In our case the easiest way to do it is to fix the coefficient of the kinetic term to be exactly 1/2 by defining

$$\phi'(\boldsymbol{x}') = \sqrt{Z_\phi}\phi^-(\boldsymbol{x}) \tag{2.50}$$

where $Z_\phi = \left(\frac{\Lambda_0}{\Lambda}\right)^{D-2} Z'_\phi$.

The final action is

$$S_\Lambda[\phi'] = \int \mathrm{d}^D x' \left(\frac{1}{2}(\boldsymbol{\nabla}'\phi')^2 + \frac{1}{2}m^2(\Lambda)\phi'^2 + \frac{\lambda_4(\Lambda)}{4!}\phi'^4 + \dots \right) \tag{2.51}$$

This whole procedure is called *RG flow*[8]. To summarise, the three steps of RG are:

1. Integrating out the high energy degrees of freedom
2. Fix the spacetime integral by rescaling the spacetime variables
3. Fix the functional integral by rescaling the fields

All these manipulations of the integrals just changed the variables of integration, therefore we can confidently write (momentarily ignoring the sources):

$$Z(\Lambda_0, g_{0,a}) = Z(\Lambda, g_a(\Lambda)) \tag{2.52}$$

Noting the LHS does not depend on Λ we can write this equation differentially:

[8] RG here stands for "renormalisation group", but this is name is completely uninformative because were are not normalising anything neither once nor twice, and also, it is not a group.

$$\Lambda \frac{\mathrm{d}Z(\Lambda, g_a(\Lambda))}{\mathrm{d}\Lambda} = \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_a \frac{\partial}{\partial g_a} \right) Z(\Lambda, g_a(\Lambda)) = 0 \tag{2.53}$$

where we substituted $\beta_a = \Lambda \frac{\partial g_a}{\partial \Lambda}$, called the β-function of the coupling g_a.

This is the first example of a Callan-Symanzik (CS) equation. It tells us that the couplings in the effective action change so as to precisely balance the change in the cut-off, and, in the end, leave everything unchanged.

Including sources will tell us what happens for correlators. We just have to be careful about redefining the sources:

$$\int \mathrm{d}^D x J^-(\boldsymbol{x}) \phi^-(\boldsymbol{x}) = \int \mathrm{d}^D x' \left(\frac{\Lambda_0}{\Lambda} \right)^D Z_\phi^{-\frac{1}{2}} J^-(\boldsymbol{x}) \phi'(\boldsymbol{x}') = \int \mathrm{d}^D x' \, J'(\boldsymbol{x}') \phi'(\boldsymbol{x}') \tag{2.54}$$

where

$$J'(\boldsymbol{x}') = \left(\frac{\Lambda_0}{\Lambda} \right)^D Z_\phi^{-\frac{1}{2}} J^-(\boldsymbol{x}) \tag{2.55}$$

Recalling (2.25) we find

$$\begin{aligned} G^{(n)}_{\Lambda_0}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_2) &= \prod_{j=1}^n \left(-\mathrm{i} \frac{\delta}{\delta J^-(\boldsymbol{x}_j)} \right) \int \mathscr{D}\phi \; \mathrm{e}^{\mathrm{i} S_{J^-}[\phi]} = \\ &= \left(\frac{\Lambda_0}{\Lambda} \right)^{nD} Z_\phi^{-\frac{n}{2}} \prod_{j=1}^n \left(-\mathrm{i} \frac{\delta}{\delta J'(\boldsymbol{x}'_j)} \right) \int \mathscr{D}\phi' \; \mathrm{e}^{\mathrm{i} S_{J'}[\phi']} \end{aligned} \tag{2.56}$$

and therefore

$$G^{(n)}_{\Lambda_0}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_2) = \left(\frac{\Lambda_0}{\Lambda} \right)^{nD} Z_\phi^{-\frac{n}{2}} G^{(n)}_\Lambda(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_2) \tag{2.57}$$

Once again noting that the LHS is independent of Λ (and being careful about the fact $\boldsymbol{x}'$ does) we can derive the differential version of this equation:

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_a \frac{\partial}{\partial g_a} + n \gamma_\phi \right) G^{(n)}_\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n; g_a(\Lambda)) = 0 \tag{2.58}$$

where

$$\gamma_\phi = -\frac{1}{2} \Lambda \frac{\partial \log Z_\phi(\Lambda)}{\partial \Lambda} \tag{2.59}$$

is a sort of β-function for the field rescaling.

This is the more traditional CS-equation. It tells us how to change the couplings (and the field normalisation) so that as we change the cut-off the final answer stays the same.

This outcome may come at a surprise. We have this very complicated looking procedure to make sure that, in the end, the final answer is the same. We might think this is all a fruitless exercise. The remaining sections of this chapter will argue the opposite. First we will just compute this flow and see how that allows us to vastly reduce the number of theories which are important at low energies, and also to look for classes of interesting theories. Then we will come back to the infinities that motivated our discussion and understand how the fact the answer is independent of the cut-off will allow us to rescue perturbation theory. Finally we will see how despite initial appearances we can create a controlled expansion of the higher order terms we neglected earlier and use that to estimate the energies at which new physics will arise.

2.3 Computing RG flows

Now it is time to get our hands dirty and calculate what happens under this RG flow. Let us start with the basics, what happens for a quadratic theory? In this case, the integration in (2.46) is trivial, it just gives a normalisation constant. We then get:

$$S_\Lambda[\phi^-] = \int_{|\boldsymbol{p}|<\Lambda} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{2} \tilde{\phi}^-(-\boldsymbol{p}) \left(\boldsymbol{p}^2 + m_0^2\right) \tilde{\phi}^-(\boldsymbol{p}) \tag{2.60}$$

Step 2 gives us:

$$S_\Lambda[\phi^-] = \int_{|\boldsymbol{p}|<\Lambda_0} \frac{\mathrm{d}^D p'}{(2\pi)^D} \left(\frac{\Lambda}{\Lambda_0}\right)^D \frac{1}{2} \tilde{\phi}^-(-\boldsymbol{p}) \left(\left(\frac{\Lambda}{\Lambda_0}\right)^2 \boldsymbol{p}'^2 + m_0^2 \right) \tilde{\phi}^-(\boldsymbol{p}) \tag{2.61}$$

hence,

$$Z_\phi = \left(\frac{\Lambda}{\Lambda_0}\right)^{D+2}, \text{ and } \tilde{\phi}'(\boldsymbol{p}') = \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{D+2}{2}} \tilde{\phi}^-(\boldsymbol{p}) \tag{2.62}$$

the final action is (dropping primes)

$$S_\Lambda[\phi] = \int_{|\boldsymbol{p}|<\Lambda_0} \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{2} \tilde{\phi}(-\boldsymbol{p}) \left(\boldsymbol{p}^2 + \left(\frac{\Lambda}{\Lambda_0}\right)^2 m_0^2 \right) \tilde{\phi}(\boldsymbol{p}) \tag{2.63}$$

We don't generate any new terms, but the mass gets rescaled as

$$m^2(\Lambda) = \left(\frac{\Lambda_0}{\Lambda}\right)^2 m_0^2 \tag{2.64}$$

or, in terms of the β-function:

$$\beta_{m^2}(\Lambda) = \Lambda \frac{\mathrm{d}m^2(\Lambda)}{\mathrm{d}\Lambda} = -2m^2(\Lambda) \tag{2.65}$$

The coefficient on the RHS is negative, which means m^2 *increases* in the IR.

Note that for $m^2 = 0$ this whole procedure has no effect. We say that this is a *fixed point*. This particular one is called the *Gaussian fixed point*.

So far this isn't very interesting, so let us add interactions. Now we would have to compute the path integral (2.46) by expanding in Feynman diagrams; but, the leading order answer is to completely neglect these diagrams and just skip to step 2. Consider a generic interactions with n powers of the field and r derivatives:

$$\int \prod_{i=1}^{n} \left(\frac{\mathrm{d}^d p_i}{(2\pi)^d} p_i^{r_i} \tilde{\phi}(\boldsymbol{p}_i) \right) \lambda_{0,n,r} \delta^{(d)} \left(\sum_{i=1}^{n} \boldsymbol{p}_i \right) \tag{2.66}$$

such that $\sum_i r_i = r$.

Then, on step 2 we get a factor of $\left(\frac{\Lambda}{\Lambda_0}\right)^{nD}$ from the measure, $\left(\frac{\Lambda_0}{\Lambda}\right)^{D}$ from the Dirac-δ, and $\left(\frac{\Lambda}{\Lambda_0}\right)^{r}$ from the derivatives. The third step gives $\left(\frac{\Lambda_0}{\Lambda}\right)^{n\frac{d+2}{2}}$. The end result is:

$$\lambda_{n,r}(\Lambda) = \left(\frac{\Lambda_0}{\Lambda}\right)^{\left(1-\frac{n}{2}\right)d+n-r} \lambda_{0,n,r} \Rightarrow \beta_{\lambda_{n,r}}(\Lambda) = -\left(\left(1-\frac{n}{2}\right)d + n - r\right)\lambda_{n,r} \tag{2.67}$$

This is similar to the mass β-function, what changes is the coefficient in front of the β-function, and especially whether it's positive or negative, Table 2.1 summarises some of the possible behaviours.

We can see that for high dimensions all interactions shrink in the IR[9]. This suggests they are not very important, even if we suspect they are there for a very high cut-off, we can always just run it a bit down and kill all interactions. With this in mind, we call couplings which shrink in the IR *irrelevant*.

For low dimensions the situation inverts. We can still add enough derivatives to make any interaction irrelevant, but there is an infinite number of interactions which grow in the

[9]Note how adding derivatives will just make the coefficient *more* positive.

	$D=1$	$D=2$	$D=3$	$D=4$	$D=5$	$D=6$
ϕ^2	-2	-2	-2	-2	-2	-2
ϕ^4	-3	-2	-1	0	1	2
ϕ^6	-4	-2	0	2	4	6
ϕ^8	-5	-2	1	4	7	10

Table 2.1 Coefficients in front of the β-functions for powers of scalar fields in various spacetime dimensions

IR. We cannot get rid of these interactions using RG so we call them *relevant*. Naively this infinite number of relevant interactions makes everything strongly coupled and thus very complicated, however, at this low number of dimensions we have a plethora of other tools available which will be beyond the scope of our treatment.

The 3- and 4-dimensional cases are somewhere in between. We have very few relevant interactions, an infinite number of irrelevant interactions, and a few which appear to not change under RG. The ones with coefficient zero we call *marginal*. To figure out their fate we would need to go to higher order in perturbation theory, then they might turn out to truly be constant, *i.e. truly* marginal, or to secretly be relevant or irrelevant, which we would call *marginally (ir)relevant*.

Before we brave going to higher order in perturbation theory to figure out the fate of these marginal interactions some points are in order.

Firstly, the coefficient in front of the β-function whose sign seems to dictate the fate of our couplings is nothing other than the mass dimension of that very coupling. We can easily check this to be true, but it is worth deriving this fact in another way so that we understand why this is true. The trick is to express everything in terms of dimensionless couplings. We define them by

$$\lambda_a = \Lambda^{[\lambda_a]} g_a \tag{2.68}$$

where square brackets denote the mass dimension. Note how we are using the current cut-off of the theory for consistency.

In these terms, the three steps of RG are[10]

$$\lambda_{0,a} = \Lambda_0^{[\lambda_a]} g_{0,a} \xrightarrow{\text{Step 1}} \lambda_a' = \Lambda^{[\lambda_a]} g_a(\Lambda) \xrightarrow{\text{Steps 2+ 3}} \lambda_a(\Lambda) = \Lambda_0^{[\lambda_a]} g_a(\Lambda) \tag{2.69}$$

[10]There is a subtlety I am sweeping under the rug. The first arrow actually should be doing part of step 2 already, the rescaling by Z_ϕ' which comes from the integrating out. The second arrow does the remaining rescaling which comes from the mass dimension of the field.

So the final steps or RG are just changing the cut-off factor back to Λ_0. In terms of the β-function, we have:

$$\beta_{\lambda_a}(\Lambda) = \Lambda \frac{\mathrm{d}\lambda_a}{\mathrm{d}\Lambda} = \Lambda_0^{[\lambda_a]} \Lambda \frac{\mathrm{d}g_a}{\mathrm{d}\Lambda} = \Lambda_0^{[\lambda_a]} \beta_a(\Lambda) \tag{2.70}$$

The upshot is that we only have to track the change in the dimensionless coupling to find out the β-function. This may seem like a huge simplification, but, in practice, sometimes it can be tricky to figure out all of the factors. In this language, neglecting step 1 means

$$\lambda_{0,a} = \lambda'_a \Leftrightarrow \Lambda^{[\lambda_a]} g_a = \Lambda_0^{[\lambda_a]} g_{0,a} \Leftrightarrow g_a(\Lambda) = \left(\frac{\Lambda_0}{\Lambda}\right)^{[\lambda_a]} g_{0,a} \tag{2.71}$$

making it obvious that

$$\beta_a = -[\lambda_n] g_a \tag{2.72}$$

which is precisely the result we had previously.

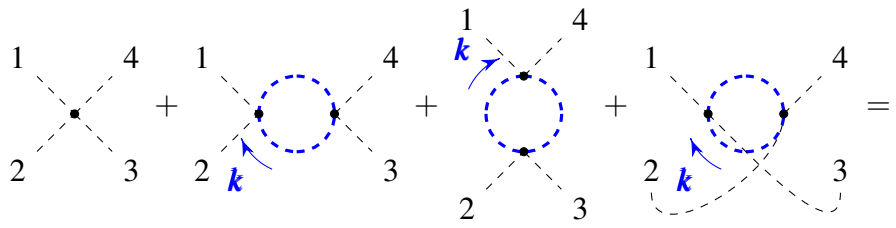
The second point worth noting is that our conclusions regarding RG flows are actually fairly generic. At an arbitrary fixed point RG flow does not do anything so the β-functions vanish. Around this point, they will generically be non-zero but we can linearise them and then diagonalise in the space of all couplings to arrive at

$$\beta_a = -\Delta_a g_a \tag{2.73}$$

where the g_a are couplings for possibly very complicated combinations of fields and their derivatives. In the vicinity of this other fixed point we can still talk of marginal, relevant and irrelevant couplings classifying if deforming in that direction will bring us away or back towards our fixed point of origin. Quite importantly, it is fairly generic that there will be an infinite number of irrelevant operators and only a handful of marginal or relevant operators (except of course in the case of $D = 1$ or 2 where we will have an infinite number of relevant operators). RG flow is highly convergent. Any random deformation will quickly converge on what is usually called the *renormalised trajectory* which only has marginal and relevant interactions.

This is one of the beauties of RG, it gives us a consistent way to get rid of the vast majority of terms we could think about and allows us to only focus on a few. This will ultimately mean that theories with very disparate origins will end up looking very similar in the deep IR which is the phenomenon of universality.

We will close off this section by calculating some 1-loop diagrams to figure out the fate of the quartic interaction in four dimensions[11] The diagrams we need to compute are[12] (we are denoting the UV modes in blue and thick lines and defining all of the external momenta to be pointing inwards):



$$
\begin{aligned}
= -\lambda_0 + \frac{\lambda_0^2}{2}\int_{\Lambda}^{\Lambda_0} \frac{\mathrm{d}^4 k}{(2\pi)^4}\Bigg(& \frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_2)^2+m_0^2} + \\
& + \frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_4)^2+m_0^2} + \\
& + \frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_3)^2+m_0^2}\Bigg)
\end{aligned}
\tag{2.74}
$$

The dependence on external momenta just give us the contribution to derivative couplings which we are not interested in, so we will set the external momenta to zero, obtaining;

$$
\begin{aligned}
& -\lambda_0 + \frac{3\lambda_0^2}{2}\int_{\Lambda}^{\Lambda_0}\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{(\boldsymbol{k}^2+m_0^2)^2} = -\lambda_0 + \lambda_0^2\frac{3\mathrm{Vol}(S^3)}{2(2\pi)^4}\int_{\Lambda}^{\Lambda_0}\mathrm{d}k\,\frac{k^3}{(k^2+m_0^2)^2} = \\
& = -\lambda_0 + \lambda_0^2\frac{3}{32\pi^2}\left(\frac{m_0^2}{m_0^2+\Lambda_0^2} - \frac{m_0^2}{m_0^2+\Lambda^2} + \log\left(\frac{m_0^2+\Lambda_0^2}{m_0^2+\Lambda^2}\right)\right)
\end{aligned}
\tag{2.75}
$$

therefore

$$
\lambda' = \lambda_0\left(1 - \lambda_0\frac{3}{32\pi^2}\left(\frac{m_0^2}{m_0^2+\Lambda_0^2} - \frac{m_0^2}{m_0^2+\Lambda^2} + \log\left(\frac{m_0^2+\Lambda_0^2}{m_0^2+\Lambda^2}\right)\right)\right) \tag{2.76}
$$

The beta function is then

[11] For three dimensions, we always have a relevant interaction which takes us to strong coupling. We need a different set of tools to examine this case [30, 34].

[12] The factors of i and/or -1 may look inconsistent with Section 2.1 but this is not a typo. The difference comes from the Wick rotation, check Appendix B from [30] for the derivation of these factors.

$$\beta_4 = \Lambda \frac{\mathrm{d}g_4}{\mathrm{d}\Lambda} = \Lambda \frac{\mathrm{d}\lambda'}{\mathrm{d}\Lambda} = \frac{3}{16\pi^2} \frac{\Lambda^4 \lambda_0^2}{(\Lambda^2 + m_0^2)^2} = \frac{3}{16\pi^2} \frac{g_4^2}{(1+g_2)^2} \tag{2.77}$$

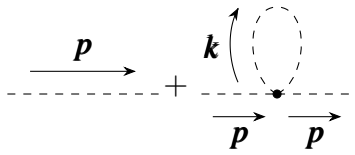
where in the last line we approximated $g_4 = \lambda_0$ and $g_2 = \Lambda^{-2} m_0^2$ which is true up to the order we are considering.

The leading contribution is positive and therefore this interaction is in fact *marginally irrelevant*. It does decay in the IR, however, the decay is incredibly slow, it decays logarithmically rather than polynomially, so it is still worth taking into account in physical models.

2.4 Connecting with observables: improving perturbation theory with RG

In the previous section we constructed the RG flow as a consistency condition on introducing an arguably unphysical cut-off. The upshot was that we can vary the cut-off as long as we compensate by changing the couplings. We then used this new tool to map out the space of available theories, understanding certain features like how most couplings are irrelevant. However, it might all seem a bit academic. How is this flow connected to observables and what happened to the infinities? Only one way to find out: calculating.

Let us start with the 2-point function. Up to 1-loop it is given by the following diagrams:

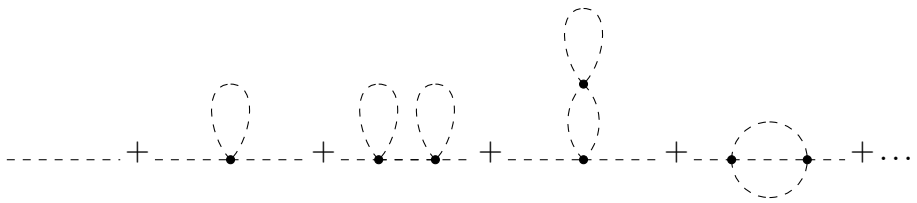


$$= \frac{1}{p^2 + m_0^2} - \left(\frac{1}{p^2 + m_0^2}\right)^2 \frac{\lambda_0}{2} \int^{\Lambda_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + m_0^2} =$$

$$= \frac{1}{p^2 + m_0^2} - \left(\frac{1}{p^2 + m_0^2}\right)^2 \frac{\lambda_0}{32\pi^2} \left(\Lambda_0^2 - m_0^2 \log\left(1 + \frac{\Lambda_0^2}{m_0^2}\right)\right) \tag{2.78}$$

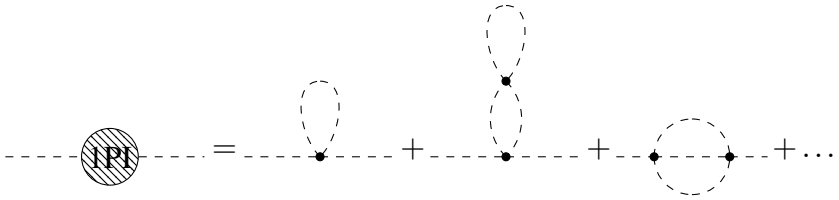
Because we have cut off the large momenta the integral is now perfectly finite, but we are not completely out of trouble yet. The whole idea was that the cut-off was much larger than whatever scales we are interested in, to avoid potential artefacts of the inherent arbitrariness of the choice of regulator. However, that means the 1-loop contribution might still be worryingly big. Introducing a cut-off only changed 'infinite' into 'big', but 'big' is still bad enough to ruin the perturbative expansion. Nevertheless, the tools from the previous section will allow us to improve perturbation theory.

To progress we first need to do some trickery to understand what these diagrams are actually computing. In general, the diagrams that contribute to the 2-point function, even beyond leading order are



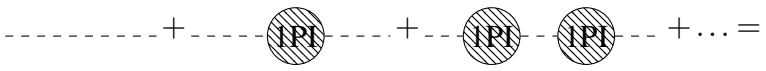
(2.79)

We see that some of the diagrams are basically copies of each other. For example, the third term looks like two copies of the second. This suggests we should be able to only consider 'distinct' diagrams. We say that a diagram is one-particle irreducible (1PI) if it has no bridges, where a bridge is a line that once removed makes the diagram disconnected. For example, the second is 1PI but the third isn't, because, if we remove the middle propagator, we get two disconnected diagrams (two copies of the second term). We can then write a series for the 1PI diagrams with two external legs:



(2.80)

We'll call this expansion $\Pi(\boldsymbol{p}^2)$ ($\boldsymbol{p}$ is the external momenta, and we have amputated the two external propagators), this is often called the *self-energy* of the field. Using this notation the propagator is then:



$$
\begin{aligned}
&= \frac{1}{\boldsymbol{p}^2+m_0^2} + \frac{1}{\boldsymbol{p}^2+m_0^2}\Pi(\boldsymbol{p}^2)\frac{1}{\boldsymbol{p}^2+m_0^2} + \frac{1}{\boldsymbol{p}^2+m_0^2}\Pi(\boldsymbol{p}^2)\frac{1}{\boldsymbol{p}^2+m_0^2}\Pi(\boldsymbol{p}^2)\frac{1}{\boldsymbol{p}^2+m^2} + \dots \\
&= \frac{1}{\boldsymbol{p}^2+m_0^2-\Pi(\boldsymbol{p}^2)}
\end{aligned}
\tag{2.81}
$$

where in going to the last line we summed the geometric series.

In this notation, the previous calculation is merely telling us that:

$$\Pi(\boldsymbol{p}^2) = -\frac{\lambda_0}{32\pi^2}\left(\Lambda_0^2 - m_0^2 \log\left(1+\frac{\Lambda_0^2}{m_0^2}\right)\right) \tag{2.82}$$

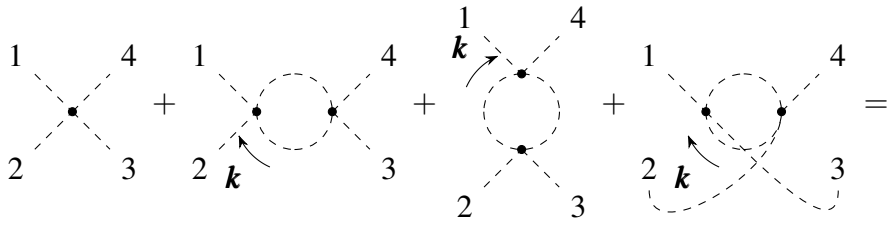
Remembering that from the Källén–Lehmann spectral representation [35, 36] we know that the physical mass is the lowest pole in the 2-point function. Therefore, because our $\Pi(\boldsymbol{p}^2)$ is independent of momenta, our calculation could be summarised by saying that there is a mismatch between the physical mass m_{phys} and the coefficient of the quadratic term in the Lagrangian m_0:

$$m_{\text{phys}}^2 = m_0^2 + \frac{\lambda_0}{32\pi^2}\left(\Lambda_0^2 - m_0^2 \log\left(1+\frac{\Lambda_0^2}{m_0^2}\right)\right) \tag{2.83}$$

And now the solution is clear. m_0 is not a holy constant that we cannot interfere with, it is just a point in the RG flow. Because of the various forms of the CS equations we know that physical quantities are invariant under the RG flow. We can flow under RG to dial down the cut-off to a Λ as close as possible to the maximum momentum we are probing. We just have to appropriately change m_0 (and all the other couplings) as we do so. The mass is a relevant coupling so it will get bigger and bigger, but we aren't treating it perturbatively so there is no immediate concern. All that matters is that our loop expansion is well behaved, and, if the cut-off is sufficiently small, it is.

The discussion is still a bit academic if we only consider the 2-point function. Choosing a small cut-off only tell us perturbation theory is valid. But then, experimentally we will just measure m_{phys}, not m_0 or Λ_0. So it is completely independent on which cut-off we choose. We need to calculate a bit more to see some non-trivial results. Let us look at the 4-point function.

The diagrams that contribute are:



$$= -\lambda_0 + \frac{\lambda_0^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_2)^2+m_0^2} + \right.$$
$$\left. + \frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_4)^2+m_0^2} + \right.$$

$$+\frac{1}{\boldsymbol{k}^2+m_0^2}\frac{1}{(\boldsymbol{k}+\boldsymbol{p}_1+\boldsymbol{p}_3)^2+m_0^2}\Bigg) \tag{2.84}$$

Where, in the first diagram, we should integrate over $\{\boldsymbol{k}|\ |\boldsymbol{k}|<\Lambda_0 \wedge |\boldsymbol{k}+\boldsymbol{p}_{12}|<\Lambda_0\}$, for $\boldsymbol{p}_{ij}=\boldsymbol{p}_i+\boldsymbol{p}_j$ because these are the momenta that appear in the propagators. Similar comments apply to the remaining diagrams.

These integrals are much harder to do because of the complicated region of integration so we won't solve it exactly. But we don't need the full expression to get the main picture. We are potentially concerned with the large Λ_0 regime. Doing some rough scaling[13], we see that we have three powers of $\boldsymbol{k}$ on top and four on the bottom. For $\Lambda_0^2 \gg m_0^2,\ \boldsymbol{p}_{ij}^2$ and in the region of large $\boldsymbol{k}$ our integrals look like

$$\int^{\Lambda_0}\frac{\mathrm{d}k}{k}\sim\log(\Lambda_0) \tag{2.85}$$

which is a slower growth but still worrying. However, we can once more choose a smaller Λ_0 (and corresponding λ_0) to tame this behaviour.

The lower end of the integral is also under control. Massaging the expression $\left|\boldsymbol{k}+\boldsymbol{p}_{ij}\right|<\Lambda_0$ we get (where $\cos\theta$ is the angle between $\boldsymbol{k}$ and $\boldsymbol{p}_{ij}$):

$$\cos\theta<\frac{\Lambda_0^2}{2\left|\boldsymbol{p}_{ij}\right||\boldsymbol{k}|}-\frac{\left|\boldsymbol{p}_{ij}\right|}{2|\boldsymbol{k}|}-\frac{|\boldsymbol{k}|}{2\left|\boldsymbol{p}_{ij}\right|} \tag{2.86}$$

From this it is straightforward to show that if $\Lambda_0<\frac{\left|\boldsymbol{p}_{ij}\right|}{2}$ (and $|\boldsymbol{k}|<\Lambda_0$) then the RHS is smaller than -1 and therefore the constraint cannot hold. In the end this means there is an effective lower bound on the integral, because the integration from 0 to $\frac{\left|\boldsymbol{p}_{ij}\right|}{2}$ vanishes.

We can only potentially have issues if the lowest $\frac{\left|\boldsymbol{p}_{ij}\right|}{2}$ is very different from the highest $|\boldsymbol{p}_i|$ because then the smallest possible cut-off would still be too big. In fact a similar issue could also arise in the 2-point function when $\boldsymbol{p}^2\gtrsim\frac{32\pi^2m_0^2}{\lambda_0}$. This is an important lesson, RG can only *improve* perturbation theory it might not *save* it. If we have an intrinsic large separation of scales we need different techniques [37, 38].

But now we have more to play with. The correlator has a somewhat complicated dependence on the external momenta. To extract the couplings we have to choose a cut-off (and we should choose the smallest possible value) and then fit our experimental data to (2.84), where m_0 is calculated by inserting the chosen cut-off value and the measured experimental mass in (2.83). If later on we end up doing experiments at a different energy

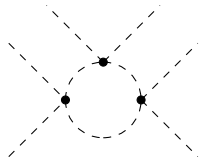
[13] A more detailed calculation of this limit can be found in [30, 32]

for which different cut-offs would be needed we can then use our RG equations to run the couplings and get a new experimental prediction. We just need to parametrise the flow once.

It is worth noting that the particular values of the couplings (and the way they depend on the cut-off) are dependent on how we chose to regulate the theory. A different choice of regulator would wield different values. We can still compare with experiment and, once the flow is parametrised, we should have exactly the same predictions for future experiments. This is all because the fundamental equations have to be the same, we would still need a CS-like equation for this all to work. At most we are just doing a coordinate transformation on the space of theories and/or a reparametrisation of the flow. These choices are like a choice of gauge. And in very much the same vein, the physical answers might be the same, but the difficulty of the calculations certainly isn't.

Before we finish this section I cannot resist just showing one more example. Let us calculate the 6-point function and see what lessons we learn. In fact, we haven't introduced a ϕ^6 coupling originally so this is a slightly different case which might be worth investigating.

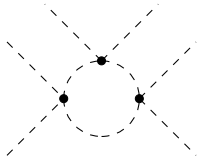
Just doing some rough scaling on the 1-loop 6-point function we see that we have 3 propagators, each giving a k^{-2}; and one integral giving a k^3, so in total



$$\sim \int^{\Lambda_0} \frac{\mathrm{d}k}{k^3} \sim \frac{1}{\Lambda_0^2} \tag{2.87}$$

which isn't 'big' at all!

This behaviour is really quite special. If we for example do this calculation in 6 dimensions rather than 4, then we have



$$\sim \int^{\Lambda_0} \frac{\mathrm{d}k}{k} \sim \log(\Lambda_0) \tag{2.88}$$

which grows logarithmically.

In principle, with our previous philosophy, despite ϕ^6 being irrelevant, we should still have introduced it and parametrised its flow. Being irrelevant just means it shrinks in the IR, but maybe it was actually really really big in the UV so right now it is still measurable. However, it seems like if we just hadn't introduced it nothing would go wrong. It seems somewhat overkill to bring the cut-off down for the 6-point correlator if it was small to begin with. At least in 4 dimensions that is

This suggest there is still some structure we are missing and, perhaps more importantly, not abusing. We need to consider more generic ways to regulate and parametrise the RG

flow and to figure out some sort of 'minimal' regulation that will not bother with regulating diagrams which were not big to begin with.

2.5 Generalising RG flows and counterterms

In the previous sections we took a constructive approach to RG. Starting from things we understand and gradually building upon them to get the structure. Now that we have used that structure to do its purpose, we can figure out what were the key elements and where were just artefacts of our starting point. In this way we shall be able to consider more generic RG flows which will help us doing calculations in practice (and understand how most people do these calculations).

The main accomplishments our construction had were turning 'infinity' into 'big', and then turning 'big' into 'small'. The way to turn 'infinity' into 'big' was via the introduction of a regulator; whereas turning 'big' into 'small' is a bit more restrictive, the key aspect being respecting equations (2.52) and (2.58) which ensure the physical observables stay the same as we mess with the regulator to improve our perturbative expansion.

Given we potentially need to change the regulator by a lot to improve perturbation theory, it's probably a bit more general to start with (2.52) rather than the differential versions.

So, what do we want? We have to start with the partition function $Z(\Lambda_0, g_{0,a})$, with a given regulator (we are calling it Λ_0 but it doesn't have to be a cut-off). This theory is perfectly finite but still has some 'big' terms in the perturbative expansion. To fix this we need to find the function $g_a(\Lambda)$ such that different choices of Λ give the same physics and that some of those choices rescues perturbation theory.

As an example, let us a look at the corrections for the physical mass (from the previous sections we now that there are no further contributions to the 2-point function at 1-loop). From the Λ_0 theory with a hard cut-off we get:

$$m^2_{\text{phys}} = m^2(\Lambda_0) + \frac{\lambda(\Lambda_0)}{2}\int^{\Lambda_0} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\boldsymbol{k}^2 + m^2(\Lambda_0)} \tag{2.89}$$

The trick to proceed is to expand the functions $m^2(\Lambda_0)$ and $\lambda(\Lambda_0)$ in powers of $\lambda_0 = \lambda(\Lambda_0)$.

$$m^2_{\text{phys}} = m_0^2 + \delta m_0^2 + \frac{\lambda_0}{2}\int^{\Lambda_0} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\boldsymbol{k}^2 + m_0^2} + O(\lambda_0^2) \tag{2.90}$$

where $m_0 = m(\Lambda_0)$ and $\delta m_0^2 = O(\lambda_0)$ is the first order term in the expansion of $m(\lambda_0)$.

This expression is still not respecting perturbation theory because the integral is still too big. So we repeat the calculation with a smaller cut-off Λ to get:

$$m^2_{\text{phys}} = m'^2 + \delta m'^2 + \frac{\lambda'}{2}\int^{\Lambda} \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\boldsymbol{k}^2 + m'^2} + O(\lambda'^2) \tag{2.91}$$

and now we match order by order[14].

At order λ_0^0 we get $m_0 = m'$.

At order λ_0^1 we get

$$\delta m'^2 - \delta m_0^2 = \frac{\lambda_0}{2}\int_{\Lambda}^{\Lambda_0} \frac{\mathrm{d}^4k}{(2\pi^4)}\frac{1}{\boldsymbol{k}^2 + m_0^2} \tag{2.92}$$

which is precisely the result we obtained earlier when integrating out the UV modes, which confirms the validity of our approach.

The biggest advantage of this matching method (in opposition to integrating out) is one we haven't exploited yet: we could use two different schemes at the two different scales. There is nothing stopping me from using a smoother cut-off or similar in (2.91), we just have to match the observables and move on, being careful to impose (2.58) with respect to Λ if we need to dial the cut-off to another relevant scale.

So now, by directly matching the observables at the two scales, we can not only use an arbitrary scheme but we can even use a different scheme at each scale. This is as general as it gets so it is time to start answering the questions from the previous section.

Note how we could only constrain the difference between the first order corrections. This behaviour is generic, it comes from the fact we could always have absorbed part of the first order terms into the definition of m_0. This freedom is equivalent to the integration constant one would get if we were to solve (2.58) instead. Because of this, in most treatments the difference is actually bundled into a single quantity δm^2, called the *counterterm*. We just have to either choose $\delta m_0^2 = 0$ or $\delta m'^2 = 0$.

In some sense, when we were integrating out, we were choosing $\delta m_0^2 = 0$. The reasoning was that we had a fixed starting point and we wanted to figure out what was the value for $m^2(\Lambda)$ (or alternatively $\delta m'^2$) that would describe the same physics.

However, to answer the questions from the previous section, we will take the alternative (and more common) approach of $\delta m'^2 = 0$. In this way we are instead asking the question: "How can we redefine the original coupling such that the 'bigness' is removed without changing the physics". Of course these two ways of formulating the question are equivalent,

[14] It is straightforward to show that, at leading order, $\lambda' = \lambda_0$.

but the re-framing is not only useful to compare with other works, but will make it easier to figure out the extra structure we were missing earlier.

And now, after all this work, it is finally time to cave in and talk of cancelling infinities. It is important to note that it was not at all necessary to speak of infinities, once we introduce the regulator everything is finite, just a bit too big for perturbation theory. However, the easiest way to check whether something is 'big' or 'small' is by taking the limit when the regulator is removed and checking whether the final answer is finite or infinite. This is in fact how counterterms and renormalisation were understood historically. It was thought we were adding pieces to the action that would 'cancel' the infinities and make everything finite and hopefully consistent. It took decades to realise there was a sort of group action at play and even more time before Wilson talked of integrating out UV modes. But now we know better. We are not cancelling infinities, we are merely imposing consistency conditions on the presence of a regulator and using that freedom to improve our perturbation theory. Nevertheless, the language stuck and it is actually a bit easier to think of 'finite' vs 'infinite' rather than 'big' vs 'small', even though that is what should be in the back of our minds.

In this language it is worthy to note that there are parts of the counterterm which are more important than others. The parts that are there to cancel the infinities are not negotiable, they are what makes everything work. However, there are always finite parts as well, these are dependent on the particular scheme, and can even be used to *define* the scheme, and so can be changed according to our will.

With this all in mind, let us revisit the 4- and 6-point functions.

Looking at (2.84) and (2.85) we can see that the $\Lambda_0 \to \infty$ limit is independent of the external momenta. As $\boldsymbol{k}$ gets big we can neglect the addition of $\boldsymbol{p}_{ij}$. So, it stands to reason that we only need to cancel the $\boldsymbol{p}_i = 0$ divergence and that will be enough to render the correlator finite for all other values of external momenta. If we keep with the cut-off language, this would mean choosing:

$$\delta\lambda = \frac{3\lambda_0^2}{2}\int_{\Lambda}^{\Lambda_0} \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{(\boldsymbol{k}^2 + m_0^2)} \tag{2.93}$$

But note the difference with what we obtained by just integrating out completely from Λ_0 to Λ. Now it won't be true that the new 4-point correlator will look like (2.84) but with Λ instead of Λ_0 everywhere. This is only true at $\boldsymbol{p}_i = 0$. In order to recover the previous result we would need to introduce counterterms for higher derivative 4-point correlators such as $\phi^2\nabla^2\phi^2$. Which is perfectly in line with what we found in section 1.3, when integrating out we do find corrections for all other couplings, they just happened to be irrelevant. What we see here is that correcting all of this is not entirely necessary, just using (2.93) is enough to

render everything finite. This is exactly what we came across in the 6-point function, it was finite to begin with, so there is no need to introduce an extra counterterm.

However, this may be a bit too hasty. We have just checked at 1-loop and for a couple of correlators. Maybe at higher loop level and/or for higher order correlators something goes wrong and we actually do need all of the counterterms to make everything consistent. It turns out that it depends. For some theories (including ϕ^4 in 4 dimensions) we can indeed only worry about a finite set of counterterms, for others (like ϕ^4 in 6 dimensions) we actually have to include an infinite number of counterterms to make everything consistent. Checking this is true at all loop orders is quite involved, but it can be summarised in the following theorem due to Bogoliubov, Parasiuk, Hepp, and Zimmermann:

Theorem (BPHZ)**.** *Quantum Field Theories can be classified into three categories:*

- ***Super-Renormalisable****: All couplings have positive mass dimension*
- ***Renormalisable****: All couplings have either positive or vanishing mass dimension*
- ***Non-Renormalisable****: At least one coupling has negative mass dimension*

These three categories will have different types of divergences and therefore their behaviour is different:

- ***Super-Renormalisable****: There are only a finite number of divergent diagrams and therefore with a finite number of counterterms we can absorb all divergences*
- ***Renormalisable****: There are an infinite number of divergent diagrams but only a finite number of divergent amplitudes. We can introduce a finite number of counterterms that can be tuned order by order to absorb all divergences. The counterterms at a given order will exactly cancel the divergent sub-diagrams at higher orders*
- ***Non-Renormalisable****: There are an infinite number of divergent amplitudes, and therefore we need an infinite number of counterterms to absorb all divergences*

And there we have it. For ϕ^4 theory in 4-dimensions we only have relevant and marginal operators so we can deal with a finite number of counterterms: δZ_ϕ, δm^2 and $\delta\lambda$. We will need to adjust them order by order but we will never need to add more. We could perfectly add more, as we did when we integrated out, but, for practical calculations, we don't have to. But this is very special to 4-dimensions, in higher dimensions the quartic coupling will be irrelevant and at sufficiently high loop/correlator order we will need to add more and more counterterms. Once we put even a single irrelevant operator we will eventually have to deal with all of them.

At first sight it can seem like theories with irrelevant operators are doomed. Given we have to add all of them it seems like we have to do an infinite number of experiments before we can calculate anything. However, an attentive reader will notice that at a given loop order we will actually only need a finite number of counterterms to fix a given amplitude. At 1-loop, no higher order coupling will be able to contribute to the 2-point function for example. So maybe there is some way to write it down such that we can do calculations, we just know that at higher precision there will be some run-away phenomenon and things will get very complicated very fast. But even then that seems optimistic. After all, we just assumed every coupling was small, but we don't necessarily have any constraints on how big g_4 is in relation to g_6, perhaps we are so unlucky that $g_6^2 \sim g_4$ and 2-loops of g_6 will come at the same order as one loop of g_4. This would mean we would have to fix all couplings before we even know which ones are important and how to write down a perturbative expansion. Very damning indeed.

These problems seemed so unsurmountable that for many decades physicists took 'non-renormalisable' as synonymous with 'inconsistent' and struggled very hard to only write down renormalisable theories. It turns out this is wrong. There is indeed a hierarchy among the different couplings and not only are non-renormalisable theories perfectly consistent they are actually far more useful than renormalisable theories.

2.6 Effective Field Theories and the origin of the cut-off

In the previous sections we have seen hints already that there should be some sort of hierarchy between the couplings. After all, we called couplings with negative mass dimension 'irrelevant' because if we flow down to the IR they would shrink very rapidly. Therefore, if, for reasons so far unknown, at a very high scale Λ_0 all couplings are comparable in size, then, at the lower scale Λ the flow under RG would dictate that irrelevant couplings will shrink, and not only that but there will be a hierarchy and higher order couplings will be smaller and smaller. So far this is wishful thinking, but in the remaining of this chapter we will understand why this behaviour is fairly generic and how to make sense of the resulting ideas.

The previous argument completely hinged on us starting at a scale Λ_0 and then lowering to a scale Λ. This may seem a bit dubious. After all, in all this time we have been working under the assumption that Λ_0 is somewhat unphysical. It was crutch, something we needed to introduce to make sense of a continuous path integral. You may think the language of infinities is not that silly in the end. If we really manage to parametrise the flow at low energies, and remove all the bigness/infinities, we can perfectly take the limit $\Lambda_0 \to \infty$.

Everything is finite and even small. No problems there. If the theory is renormalisable at least.

But is this really the case? After all, ϕ^4 is marginally *irrelevant*. Sure we can introduce counterterms, but in the end, it is just like any other irrelevant coupling, it just changes way more slowly. We can even solve the β-function at $m = 0$ to see this more clearly:

$$\beta_4 = \frac{3g_4^2}{16\pi^2} \Rightarrow g_4(\Lambda) = \frac{16\pi^2}{3\log\left(\frac{\Lambda_\infty}{\Lambda}\right)} \tag{2.94}$$

At the scale Λ_∞ the coupling becomes infinite! This phenomenon is called a Landau pole and it signals that perturbation theory breaks down in a fundamental way at exponentially high energy scales. We can only see this because by solving the β-function we are in fact re-summing part of the perturbative expansion [39].

So it seems like this theory actually doesn't make sense as a continuum theory. At some point it becomes non-perturbative and we would have to change our approach. The fact we could take $\Lambda_0 \to \infty$ was a mere artefact of perturbation theory. And if the theory doesn't have a good continuum limit (or at least not one we know how to describe) then maybe there is an actual honest-to-goodness *physical* cut-off, Λ_{EFT} which in the previous case would be (at least close to) Λ_∞. We then say we are dealing with an 'Effective Field Theory' (EFT for short) which is only valid below a certain energy scale.

We have managed to argue for the existence of at least *some* physical cut-offs, but we still don't have a hierarchy. Who knows the relative size of all the couplings at this very complicated scale Λ_{EFT} when our perturbative description breaks down.

In general, this is all we have got. Theory space is very wild, all sorts of things might happen. But actually, it seems like we have fairly generic reasons to be able to assume all $g_a \sim O(1)$ at the EFT scale. We will not attempt to prove this statement, we will merely show a very simple example which exhibits this behaviour to give intuition as to how this can appear, we defer to more comprehensive treatments of EFTs like [37, 38] for further details.

Let us consider the following very simple theory with two scalar fields ϕ and χ such that one is much heavier than the other $m_\chi \gg m_\phi$ and with a dimensionless interaction between the two:

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\boldsymbol{\nabla}\chi)^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \right) \tag{2.95}$$

If we are only probing energies far below m_χ then it stands to reason we can integrate out χ completely given we cannot hope to create such a particle in a scattering experiment. Doing so will involve very similar diagrams to Section 2.3 but instead of a UV ϕ^+ mode

with momentum between Λ and Λ_0 we use the very massive field χ. But the logic is the same: insertions of ϕ essentially become sources and χ is the only field allowed to propagate because that is what we are integrating over. Using the same logic we will generate a theory for the field ϕ, which will have all the higher order interactions, and which we only trust for energies far below m_χ.

It is a straightforward exercise to compute the new ϕ interactions. We are only interested in what happens for the irrelevant interactions so we won't bother with regulating the theory given those integrals are finite. We are pretending this is a fundamental theory with a well defined continuum limit. Computing the first couple irrelevant terms we get:

$$\nabla^2\phi^4 : \frac{3\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{3\boldsymbol{k}^2-m_\chi^2}{(\boldsymbol{k}^2+m_\chi^2)^4} = \frac{5\lambda^2}{64\pi^2 m_\chi^2} \sim 0.008\lambda^2 m_\chi^{-2} \tag{2.96}$$

$$\nabla^4\phi^4 : \frac{3\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{5\boldsymbol{k}^4-10\boldsymbol{k}^2m_\chi^2+m_\chi^4}{(\boldsymbol{k}^2+m_\chi^2)^6} = \frac{\lambda^2}{320\pi^2 m_\chi^4} \sim 0.0003\lambda^2 m_\chi^{-4} \tag{2.97}$$

$$\phi^6 : \frac{15\lambda^3}{8}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(\boldsymbol{k}^2+m_\chi^2)^3} = \frac{15\lambda^3}{256\pi^2 m_\chi^2} \sim 0.006\lambda^3 m_\chi^{-2} \tag{2.98}$$

Even though the couplings aren't exactly $O(1)$ we see that it is not far off. The important features is that higher powers of ϕ mean higher powers of λ and lower mass dimension mean lower powers of m_χ. The numerical coefficients multiplying this are all small and comparable between one another.

When now computing observables in the ϕ theory we should restrict to energies much smaller than m_χ. So we have an effective physical cut-off: $\Lambda_{\mathrm{EFT}} = m_\chi$. When we do RG to improve perturbation theory we will introduce a lower cut-off Λ, which means the dimensionless couplings g_a will all get multiplied by $\left(\frac{\Lambda}{m_\chi}\right)^{[g_a]}$, which we have assumed is small. We have our hierarchy: as a coupling becomes more irrelevant it gets suppressed by higher and higher powers of $\frac{\Lambda}{m_\chi} \ll 1$.

And this is the EFT trick. From the bottom-up perspective, we assume there is a scale Λ_{EFT} much higher than our scales of interest at which the irrelevant couplings are similar in size. Then, we run the RG game to improve our perturbation theory and fix the couplings at the energies we are interested in. Because we have irrelevant couplings we will need to introduce all of them. However, if they have a lower mass dimension they will received higher powers of $\frac{\Lambda}{\Lambda_{\mathrm{EFT}}} \ll 1$. So we have a hierarchy between them. Yes we have to introduce all of them in principle, but we can truncate this in practice and only consider them in turn. It is just a matter of being careful about when do particular couplings come into play, but

we have a controlled expansion to play with. Doing this in detail is beyond the scope of this chapter but you can find good practical treatments in [37, 38].

But, before we finish, note how this expansion only works if $\Lambda \ll \Lambda_{\text{EFT}}$. Interestingly, as we approach this scale, all the different couplings will become more similar and we will break our approximation. At $\Lambda \sim \Lambda_{\text{EFT}}$ our theory has become non-perturbative.

Conversely, from the bottom-up perspective, we can use that fact to estimate Λ_{EFT}. As soon as we have enough precision to be able to measure the first irrelevant coupling we can use that information to then estimate the energy scale at which they would become non-perturbative[15], which is our estimate for Λ_{EFT}. This scale, as we have seen could be from a very massive field that we haven't included or some other type of new physics.

And these techniques work! Historically we had predictions for the mass of the W and Higgs bosons way before we actually managed to produce them. By measuring the irrelevant couplings left behind we could be certain new physics was on the horizon.

In some sense, renormalisable theories are *less* predictive than non-renormalisable ones because it is so much harder to calculate when new physics might arise. The extra structure needed to make non-renormalisable theories work gives us a plethora of information crucial to predict when new physics might arise. Even though technically we could already do this for marginally irrelevant couplings. The difference is that when we measure an irrelevant coupling Λ_{EFT} will be polynomially higher than our experimental scales rather than exponentially higher.

And there we have it. From regulating nasty infinities away. To carving out flows in the space of theories to make that regulator consistent. Then using said flows to recover perturbation theory. And finally unveiling the structure of effective field theories which can use irrelevant couplings to predict new particles before we even produce them. These are the fruits were can bear from understanding the interplay of different scales in quantum field theory. Nevertheless, in the quest for simplicity, we have left out a very important player: symmetries! The role of symmetries cannot be understated, they will definitely make appearances in the following chapters. But, understanding their role both in isolation and in their interplay with scales is a story for another time.

[15] A very convenient way to estimate these energies is via the optical theorem [40]

Chapter 3

Developing local RG and testing quantum RG with the BFSS model

Now that we have a better understanding of the basics of RG it is only reasonable to push the limits of our understanding. In this chapter we will focus on the tantalising possibility that via the AdS/CFT correspondence we can somehow geometrise the RG flow of certain quantum field theories. To explore this possibility we will focus on the BFSS model of N coincident D0-branes, this model is especially simple due to the fact it is 1-dimensional.

Firstly, as a warm-up, we perform standard Wilsonian RG, investigating the conditions under which supersymmetry is preserved along the flow. Next, we develop a local RG scheme such that the cutoff is spacetime dependent, which could have further applications to studying QFT in curved spacetime. Finally, we test the conjecture put forward in [41] that the method of quantum RG could be the mechanism responsible for the gauge/gravity duality by applying it to the BFSS model, which has a known gravitational dual. Although not entirely conclusive some questions are raised about the applicability of quantum RG as a description of the AdS/CFT correspondence.

Throughout the chapter the relevant models will be introduced to the depth needed to follow the arguments, a brief introduction of the aspects of AdS/CFT relevant to the discussion is also presented.

This chapter is based on [1] written in collaboration with Jorge E. Santos.

3.1 Introduction

In its most precise form, the AdS/CFT correspondence is an equality of partition functions, where sources in the field theory side correspond to boundary conditions on the dynamical

fields of the gravity side [25–28]. In the large N limit on the field theory side, and in the classical limit on the gravity side, we get, roughly,

$$\left\langle \exp\left(\int \mathrm{d}^d x \mathscr{O}\phi^{(0)}\right)\right\rangle_{\mathrm{QFT}} = \mathrm{e}^{-S_{\mathrm{SuGra}}}\Big|_{\lim_{z\to 0}\bar{\phi}(z,x)z^{\Delta-d}=\phi^{(0)}(x)} . \tag{3.1}$$

We can then use this to calculate correlation functions on both sides. However, this is not the whole story. If we try to evaluate the classical action as it stands, with boundary conditions precisely on the boundary of AdS, we would get infinity. As is standard in QFT calculations, the way to deal with this infinity is to do renormalisation, *i.e.* introducing counterterms to absorb the infinities. This procedure has been extensively developed, and is now a very standard technique under the name of Holographic Renormalisation [42–46].

There are many interesting peculiarities with this idea. Firstly, it seems that what would normally be the UV divergences in standard QFT are in fact IR divergences in the gravity side. Further, what plays the role of renormalisation scale is in fact the radial direction in AdS spacetime. This is but one of the many hints that there is some deep connection between scale in the field theory side, and the radial direction in the gravity side [42, 47].

Nonetheless, despite it's success, this also leaves many questions unanswered. The most immediate one is diffeomorphism invariance. What do we mean by radial direction? That is surely not a gauge invariant statement. Secondly, it's now very well known that renormalisation in QFT is not about removing annoying infinities, it's about coarse graining, integrating out degrees of freedom we do not have access to, in order to get a description relevant at our desired scale [48]. Is there any way we can understand Holographic Renormalisation under a Wilsonian point of view?

As is to be expected, these questions have long been explored. It didn't take very long to understand that the would be RG flow in the gravity side is given by the Hamilton-Jacobi formulation, where instead of time evolution we consider radial evolution [49–51]. There have also been many proposals on how to give a more diffeomorphic invariant meaning to this radial direction [52–56]. Most of which include interpreting different RG schemes on the QFT side as different coordinate systems in the bulk. What was in general poorly understood is which scheme corresponds to which coordinate system. More recently there is a proposal for the generic correspondence between smooth schemes on both sides [57], and another for the particular case of dimensional regularisation [58]. As is to be expected, the relation between the two is not at all simple.

The difficulty with all these ideas (and the major interest) lies in the fact that, in order to get a full understanding of this issue, we would need to perform some sort of RG on the field theory side, and then compare with some sort of radial evolution on the gravity side,

which, essentially, requires proving the conjecture. Conversely, we could also go the other way around, instead of thinking that it's a shame we need to prove the conjecture in order to answer these questions, we can try to answer these questions as a means to try to prove the conjecture. The goal of this chapter is to do precisely that, to test, in a simple case, whether one of these proposals holds or not.

The proposal to be analysed in particular is the Quantum Renormalisation Group (QRG) [41, 59, 60], which, briefly, consists of applying the following procedure to a QFT with matrix valued fields (this will be covered in more detail in section 3.4.1):

1. Turn on single trace operator deformations with sources
2. Do an infinitesimal local RG transformation
3. Add auxiliary dynamical fields to project onto the space of single trace operators
4. Iterate

In this way, from a d-dim QFT we generate a $(d+1)$-dim action where what were sources are now dynamical fields. The proposal in [41] is that the new action would be the holographic dual to that CFT giving a concrete realisation of the AdS/CFT correspondence.[1]

Since the original paper, some follow-up work has been done, namely some hints for it's application to the original AdS_5/CFT_4 case [71], a concrete calculation for the $U(N)$ vector model [72], and, understanding the conditions under which one can recover full $(d+1)$-dim diffeomorphism invariance [73]. However, there has been no explicit calculation, starting from a QFT with a known gravitation dual, performing QRG and checking whether we end up with the same theory.

This is exactly what has been accomplished in this chapter. The QFT chosen was the $\mathscr{N} = 16$, one-dimensional super Yang-Mills theory with gauge group $SU(N)$, more commonly known as the BFSS model after the authors of [74]. This theory not only has a known gravitational dual [75–78], but also is extremely simple given that it is one-dimensional, a fact which allows us to perform all calculations explicitly. In the end after we perform QRG the results seem to differ from the gravity predictions [76, 79] (which have matched by lattice simulations [80]). Even though QRG cannot be completely ruled out some questions are raised as to what would be needed to make it work or prove it wrong.

We begin section 3.2 by performing standard (*i.e.* not quantum) RG on the BFSS model. This result by itself, as far as the authors are aware, is absent from the literature, mainly because there are no UV divergences, therefore, by itself, this is not very useful.

[1] Other similar proposals include [61–70], however, this chapter will restrict its attention to testing QRG.

However, it turns out it is a very useful playground to explore how one can break or preserve supersymmetry under an RG flow since we can compute everything explicitly. In section 3.3, we address the first main concern, how to define a local version of RG. It turns out one can define this under certain restrictions, and we give a concrete example of how to achieve this. We have developed this formalism to apply to QRG, however, it may be interesting in it's own right, *e.g.* if one wanted to perform RG in a curved background spacetime. Finally, in section 3.4, we put everything together and perform QRG on the BFSS model. We start be reviewing the QRG procedure in detail and the holographic duality in BFSS. Then we go to the main calculations, highlighting the disagreement with known results.

3.2 Renormalisation Group flow of BFSS model

In this section we calculate the renormalisation group flow of the BFSS model in the case where the renormalisation scale is spacetime independent. We start by a brief review of the BFSS model, then we move on to the calculation using a hard momentum cutoff. Already here we find interesting ways to avoid breaking supersymmetry. After this prelude we discuss how to implement RG with a smooth cutoff in the sense of exact RG, we find that we always break supersymmetry in that case. Finally we give some remarks on (failed) attempts to circumvent the aforementioned supersymmetry breaking.

3.2.1 Overview of the model

The BFSS model is the maximally supersymmetry matrix quantum mechanics describing the dynamics of N D0-branes. Equivalently, it is the $\mathscr{N}=16$ super Yang-Mills theory in $d=1$ dimensions with gauge group $SU(N)$, which can be obtain by dimensional reduction of the $\mathscr{N}=1$ super Yang-Mills in $d=10$ dimensions. It was originally introduced in [74] as a description of M-theory in the infinite momentum frame in the uncompactified limit, only later was its role in the gauge/gravity duality fully appreciated [75]. For a general review of this model see [81].

This theory has an $SU(N)$ gauge field A, nine scalars X_i ($i=1,\dots,9$), and 16 fermions ψ_α ($\alpha=1,\dots,16$). Both the scalars and the fermions are in the adjoint representation of the gauge group and therefore are represented by Hermitian, traceless, $N\times N$ matrices. The action for this model is (in Euclidean time):

$$S[A,X,\psi]=\frac{N}{\lambda}\int \mathrm{d}\tau\,\mathrm{Tr}\left\{\frac{1}{2}(\mathrm{D}_\tau X_i)^2+\frac{1}{2}\psi_\alpha \mathrm{D}_\tau\psi_\alpha+\right.$$

$$+\frac{1}{2}\psi_\alpha(\gamma_i)_{\alpha\beta}[X_i,\psi_\beta]-\frac{1}{4}[X_i,X_j]^2\Bigg\} \tag{3.2}$$

where $\lambda = Ng_{\rm YM}^2$ is the usual 't Hooft coupling. We are using the convention where the generators of the Lie algebra are Hermitian, and therefore they obey

$$[T^a,T^b] = \mathrm{i} f_{abc}T^c\,. \tag{3.3}$$

Furthermore, we normalise T as $\mathrm{Tr}\big(T^aT^b\big) = \delta^{ab}$. The covariant derivative in Eq. (3.2) acts as $\mathrm{D}_\tau = \partial_\tau + \mathrm{i}[A,\cdot]$. Finally, γ_i are the nine-dimensional Dirac gamma matrices, which are real, symmetric matrices satisfying $\{\gamma_i,\gamma_j\} = 2\delta_{ij}$.

As mentioned above, this theory is invariant under a supersymmetry transformation with 16 supercharges whose precise form will not be relevant for the subsequent discussion. Note also that the gauge field is not dynamical, therefore we can completely fix the gauge with $A = 0$ without the need to introduce Fadeev-Popov ghosts. This is one of the many simplifying aspects of the theory. In the remainder of the manuscript we assume we are in such a gauge.

It will also prove to be useful to do the rescaling,

$$\tilde{X}_i = \sqrt{\frac{N}{\lambda}}X_i, \qquad \tilde{\psi}_\alpha = \sqrt{\frac{N}{\lambda}}\psi_\alpha \tag{3.4}$$

so that, in these new variables, the action looks like,

$$S[X,\psi] = \int \mathrm{d}\tau\,\mathrm{Tr}\Bigg\{\frac{1}{2}(\partial_\tau\tilde{X}_i)^2+\frac{1}{2}\tilde{\psi}_\alpha\partial_\tau\tilde{\psi}_\alpha+ \\ +\frac{1}{2}\sqrt{\frac{\lambda}{N}}\tilde{\psi}_\alpha(\gamma_i)_{\alpha\beta}[\tilde{X}_i,\tilde{\psi}_\beta]-\frac{\lambda}{4N}[\tilde{X}_i,\tilde{X}_j]^2\Bigg\}\,. \tag{3.5}$$

We note that in the large N limit $N\to\infty$, the original untilded variables are $\mathscr{O}(N^0)$. Finally, in order to do the perturbative calculations presented in the subsequent sections, it is convenient to write the action in terms of the structure constants,

$$S[X,\psi] = \int \mathrm{d}\tau\Bigg\{\frac{1}{2}(\partial_\tau\tilde{X}_i^a)^2+\frac{1}{2}\tilde{\psi}_\alpha^a\partial_\tau\tilde{\psi}_\alpha^a+\mathrm{i}\frac{1}{2}\sqrt{\frac{\lambda}{N}}(\gamma_i)_{\alpha\beta}f_{abc}\tilde{\psi}_\alpha^a\tilde{X}_i^b\tilde{\psi}_\beta^c+ \\ +\frac{\lambda}{4N}f_{abe}f_{cde}\tilde{X}_i^a\tilde{X}_j^b\tilde{X}_i^c\tilde{X}_j^d\Bigg\}\,. \tag{3.6}$$

From Eq.(3.6) we can easily read the associated Feynman rules:

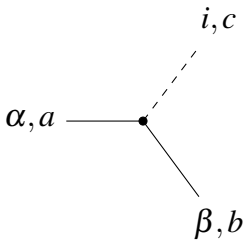
- Scalar propagator:

$$i,a \xrightarrow{\quad p \quad} j,b = \frac{\delta_{ij}\delta_{ab}}{p^2}$$

- Fermion propagator:

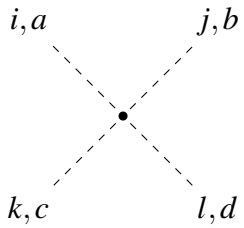
$$\alpha,a \xrightarrow{\quad p \quad} \beta,b = \frac{\delta_{\alpha\beta}\delta_{ab}}{p}$$

- Cubic coupling:



$$= -\mathrm{i}\sqrt{\frac{\lambda}{N}}(\gamma_i)_{\alpha\beta} f_{cba}$$

- Quartic coupling:



$$\begin{aligned} = -\frac{\lambda}{N}\big[\quad & f_{abe}f_{cde}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \\ + \quad & f_{ace}f_{bde}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}) + \\ + \quad & f_{ade}f_{bce}(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\big] \end{aligned}$$

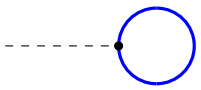
3.2.2 RG with a hard momentum cutoff

As a warm-up calculation, we start by computing the perturbative 1-loop RG flow of this model. Since this is a one-dimensional theory, there will be an infinite number of relevant interactions that will be turned on by the RG flow, rendering our perturbative approximation useless. We will, nonetheless, proceed with the calculations and only consider diagrams with up to four external legs. This is completely artificial and unjustified, however, we will proceed with this calculation because there are still some interesting lessons to take from this analysis to do with supersymmetry.

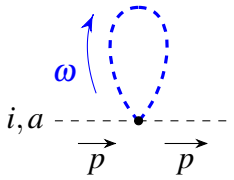
We will impose a hard momentum cutoff by demanding that our fields only have support for momenta $|p| < \Lambda_0$. Then, to lower the cutoff, we integrate over modes with support in momentum space $\Lambda < |p| < \Lambda_0$. The calculations themselves involve rather tedious index manipulations, for that reason we relegate the details to the appendix in the original paper [1]

and only present the main results here. The relevant diagrams at 1-loop order and up to four external fields are (where we denote the high energy modes with blue and thick lines):

Tadpole This one is trivially zero by the index structure.

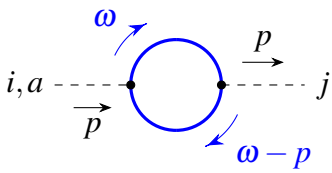


Scalar propagator There are two diagrams that contribute. We can either have scalar loop



$$= -16\lambda\,\delta_{ab}\delta_{ij}\int_{|\omega|\in[\Lambda,\Lambda_0]}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega^2} = -\frac{16\lambda}{\pi}\delta_{ij}\delta_{ab}\left(\frac{1}{\Lambda}-\frac{1}{\Lambda_0}\right),$$

or a fermionic loop



$$= 16\lambda\,\delta_{ab}\delta_{ij}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega(\omega-p)}\,. \tag{3.7}$$

For the scalar mode we must have $|\omega|\in[\Lambda,\Lambda_0]$. For the fermionic mode, one might naively think that the region of integration is also $|\omega|\in[\Lambda,\Lambda_0]$, just as for the scalar. However, that would be wrong. In fact there is also a high energy mode with momentum $\omega-p$ so, since that mode only has support when *its* momentum is in the range $[\Lambda,\Lambda_0]$ we must also impose that $|\omega-p|\in[\Lambda,\Lambda_0]$. Usually, integrating over these intricate regions is prohibitively difficult, however, for one-dimensional integrals, they can be done analytically. If we do not integrate over this region, we get non-sensical answers. For instance, the answer would depend on which line of the loop we give momentum ω and which line we give momentum $\omega-p$[2].

Let us define

$$I\equiv\{\omega|\ |\omega|\in[\Lambda,\Lambda_0]\wedge|\omega-p|\in[\Lambda,\Lambda_0]\} \tag{3.8}$$

[2]For this diagram that does not happen because the two lines are identical, but further ahead one we would see such an effect.

which brings Eq. (3.7) to

$$16\lambda\,\delta_{ab}\delta_{ij}\int_I \frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega(\omega-p)} = \begin{cases} \dfrac{16\lambda\,\delta_{ij}\delta_{ab}}{p\pi}\log\left[\dfrac{(\Lambda+p)(\Lambda_0-p)}{\Lambda\Lambda_0}\right], & p>0 \\ \dfrac{16\lambda\,\delta_{ij}\delta_{ab}}{p\pi}\log\left[\dfrac{\Lambda\Lambda_0}{(\Lambda-p)(\Lambda_0+p)}\right], & p<0 \end{cases}.$$

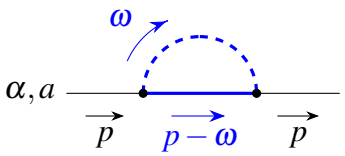
Expanding in powers of p yields

$$\frac{16\lambda\,\delta_{ij}\delta_{ab}}{\pi}\left[\left(\frac{1}{\Lambda}-\frac{1}{\Lambda_0}\right)-\frac{|p|}{2}\left(\frac{1}{\Lambda_0^2}+\frac{1}{\Lambda^2}\right)+\frac{p^2}{3}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right)+O(p^3)\right].$$

There is a linear term in p which could be worrisome, however, in $d=1$ this is a total derivative, so we shall drop it. Note that the would-be mass term cancels between the two diagrams and we are left with just a wavefunction renormalisation contribution.

$$Z_X' = 1-\frac{16\lambda}{3\pi}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right). \tag{3.9}$$

Fermion propagator There is only one diagram that contributes:



$$= 18\lambda\,\delta_{ab}\delta_{\alpha\beta}\int_I \frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega^2(p-\omega)}$$

where once again we have to be careful about the integration region and integrate over I as defined in Eq. (3.8):

$$\begin{cases} \dfrac{9\lambda\,\delta_{ab}\delta_{\alpha\beta}}{p^2\pi}\left[p\left(\frac{1}{\Lambda}-\frac{1}{\Lambda_0}+\frac{1}{\Lambda+p}-\frac{1}{\Lambda_0-p}\right)+2\log\left(\frac{\Lambda\Lambda_0}{(\Lambda+p)(\Lambda_0-p)}\right)\right], & p>0 \\ \dfrac{9\lambda\,\delta_{ab}\delta_{\alpha\beta}}{p^2\pi}\left[p\left(\frac{1}{\Lambda_0}-\frac{1}{\Lambda}+\frac{1}{\Lambda_0+p}-\frac{1}{\Lambda-p}\right)+2\log\left(\frac{\Lambda\Lambda_0}{(\Lambda_0+p)(\Lambda-p)}\right)\right], & p<0 \end{cases}. \tag{3.10}$$

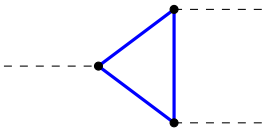
Expanding in powers of p we get,

$$\frac{3\lambda\,\delta_{ab}\delta_{\alpha\beta}}{\pi}|p|\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right), \tag{3.11}$$

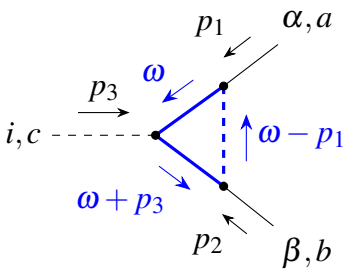
which gives a wavefunction renormalisation of

$$Z'_\psi = 1 - \frac{3\lambda}{\pi}\left(\frac{1}{\Lambda^3} - \frac{1}{\Lambda_0^3}\right). \tag{3.12}$$

Triangle diagram This is also trivially zero by the index structure



Cubic coupling There is only one diagram that contributes

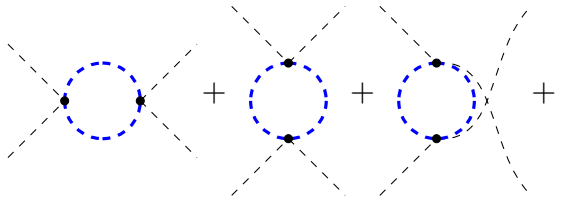


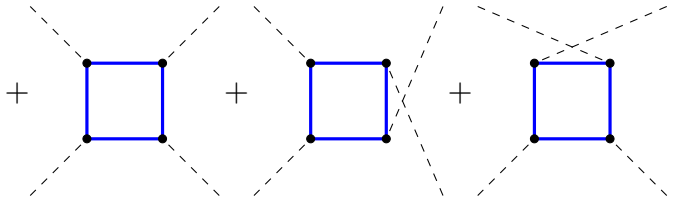
$$= 7\mathrm{i}\lambda\sqrt{\frac{\lambda}{N}}(\gamma_i)_{\alpha\beta} f_{acb} \int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\omega(\omega+p_3)(\omega-p_1)^2}\,.$$

Since we just want the correction to the cubic coupling we will set the external momenta to zero. This also means there are no subtleties with the region of integration. We then get for the correction to the cubic coupling,

$$\lambda'_{(3)} = \sqrt{\frac{\lambda}{N}}\left[1 - 7\lambda \int \frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega^4}\right] = \sqrt{\frac{\lambda}{N}}\left[1 - \frac{7}{3\pi}\lambda\left(\frac{1}{\Lambda^3} - \frac{1}{\Lambda_0^3}\right)\right]. \tag{3.13}$$

Quartic coupling Now there are six diagrams that contribute at 1-loop order, they are all distinct and rather messy. However, setting the external momenta to zero allows us to add up all these diagrams to get something nice in the end. After the dust settles the correction to the quartic couplic is:





$$\lambda'_{(4)} = \frac{\lambda}{N}\left[1+\frac{4\lambda}{3\pi}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right)\right]. \tag{3.14}$$

Putting everything together, that is, including the wavefunction renormalisation and classical scaling into account we find, to leading order in λ,

$$\lambda_{(3)}(\Lambda) = \sqrt{\frac{\Lambda_0}{\Lambda}}\sqrt{\frac{\lambda}{N}}\left[1+\frac{10\lambda}{3\pi}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right)\right] \tag{3.15}$$

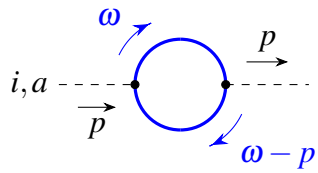
$$\lambda_{(4)}(\Lambda) = \frac{\Lambda_0}{\Lambda}\frac{\lambda}{N}\left[1+\frac{12\lambda}{\pi}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right)\right]. \tag{3.16}$$

Even though we have not generated anything as egregious as a mass term for either the fermions or the scalars, the contribution to the cubic and quartic couplings is not quite right. At the quantum level, with this regulator, $\lambda_{(4)} \neq \lambda_{(3)}^2$ which signals a breaking of supersymmetry.

By themselves, these results are not very surprising. In this theory, the supersymmetry algebra only closes on-shell, so a hard momentum cutoff will necessarily break supersymmetry (the next section will delve deeper into this issue). However, we have noticed a somewhat bizarre feature for which the interpretation is still not entirely clear (which is the main reason for including these calculations in the final manuscript). We can preserve supersymmetry at the 1-loop level if we prescribe the integration in a slightly different way. Instead of integrating with the physical constraint that all internal lines are high energy, we tried using the Feynman parameter method, which is usually used to combine propagators and make integrals more tractable (in our case we can do the calculation in both ways and compare the final answer). We then impose that the final integral is the one that sits in the range $[\Lambda,\Lambda_0]$. Like we previously mentioned, this is physically rather dubious, but it corresponds to the standard practice in higher dimensions (see for instance [82]), and, surprisingly enough, it appears to preserve supersymmetry.

The only diagrams that change are the contribution to the scalar propagator with a fermionic loop and the fermionic propagator. The contribution to the scalar propagator with

a fermionic loop now yields

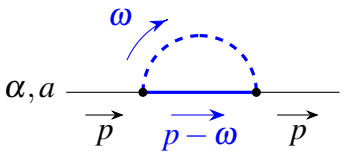


$$= 16\lambda\,\delta_{ab}\delta_{ij}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega(\omega-p)} =$$

$$\begin{aligned}
&= 16\lambda\,\delta_{ab}\delta_{ij}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{(\omega-xp)^2} = \\
&= 16\lambda\,\delta_{ab}\delta_{ij}\int_0^1 \mathrm{d}x\int_{|l|\in[\Lambda,\Lambda_0]}\frac{\mathrm{d}l}{2\pi}\frac{1}{l^2} = \\
&= \frac{16\lambda}{\pi}\delta_{ij}\delta_{ab}\left(\frac{1}{\Lambda}-\frac{1}{\Lambda_0}\right),
\end{aligned} \tag{3.17}$$

which precisely cancels the contribution from the scalar loop, meaning there is no scalar wavefunction renormalisation with this regulator.

Finally the fermionic propagator becomes,



$$= 18\lambda\,\delta_{ab}\delta_{\alpha\beta}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega^2(p-\omega)} =$$

$$\begin{aligned}
&= 18\lambda\,\delta_{ab}\delta_{\alpha\beta}\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}\omega}{2\pi}\frac{p-\omega}{\left[(\omega-xp)^2+x(1-x)p^2\right]^2} = \\
&= 18\lambda\,\delta_{ab}\delta_{\alpha\beta}\int_0^1 \mathrm{d}x\int_{|l|\in[\Lambda,\Lambda_0]}\frac{\mathrm{d}l}{2\pi}\frac{-l+(1-x)p}{\left[l^2+x(1-x)p^2\right]^2} = \\
&= 18\lambda\,\delta_{ab}\delta_{\alpha\beta}\int_0^1 \mathrm{d}x\int_{|l|\in[\Lambda,\Lambda_0]}\left(-\frac{1}{l^3}+\frac{(1-x)p}{l^4}+O(p^2)\right) = \\
&= \frac{3\lambda\,\delta_{ab}\delta_{\alpha\beta}}{\pi}p\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right),
\end{aligned} \tag{3.18}$$

which is exactly the same result as before.

Putting everything together we get,

$$\lambda_{(3)}(\Lambda) = \sqrt{\frac{\Lambda_0}{\Lambda}}\sqrt{\frac{\lambda}{N}}\left[1+\frac{2\lambda}{3\pi}\left(\frac{1}{\Lambda^3}-\frac{1}{\Lambda_0^3}\right)\right] \tag{3.19}$$

$$\lambda_{(4)}(\Lambda) = \frac{\Lambda_0}{\Lambda}\frac{\lambda}{N}\left[1 + \frac{4\lambda}{3\pi}\left(\frac{1}{\Lambda^3} - \frac{1}{\Lambda_0^3}\right)\right], \tag{3.20}$$

which now preserves supersymmetry at the quantum level.

Therefore, we have found a regulator that indeed preserves supersymmetry at least at 1-loop level. However, the physical interpretation of this regulator is not at all clear, and it does not seem to be usable beyond perturbation theory. Nevertheless, it would be interesting to see if similar phenomena occur for other theories in higher dimensions. We will not pursue this further in this manuscript, leaving it to future work.

3.2.3 RG with smooth regulators

As we mentioned in the introduction, the last calculation was mostly a warm-up calculation before doing full quantum RG. However, in order to have a local notion of scale we cannot impose a cutoff in Fourier space. Indeed, if the cutoff depends on spacetime, the Fourier transform is no longer invertible[3]. Therefore we need to use a smoother procedure. To that effect, we will use some basic exact RG technology to implement a smooth cutoff. We shall remain in momentum space for this section for convenience, in section 3.3.1 we address how to extend this to position space. We only need the most basic ideas of exact RG, nonetheless, we will review them for completeness. We closely follow the derivation in the beginning of [83], for some other reviews on the topic of exact RG you can refer to [84–88].

Let us consider scalar field theory for illustration. The key idea is to introduce a function $K(x)$ such that:

1. $K(x)$ is a smooth, non-increasing, positive function of x;
2. $K(x) = 1$ for $x < 1$;
3. $K(x) \to 0$ as $x \to \infty$ sufficiently fast.

See for example Fig (3.1) for a function satisfying all the above criteria.

These requirements can be satisfied by a smooth function, however, no analytic function works. Nevertheless, we can soften the second requirement, and only impose that $K(0) = 1$ and that $K(x)$ is suitably close to 1 for $x < 1$. Then, we can find suitable analytic functions,

[3]This is quite easy to see. For example, take some function $f(x)$, the normal Fourier transform with a cutoff would give you $f(x) = \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}kx} \hat{f}(k)$, and we can easily check that indeed this is invertible: $\hat{f}(k) = \int_{\mathbb{R}} \mathrm{d}x \mathrm{e}^{-\mathrm{i}kx} f(x) = \int_{\mathbb{R}} \mathrm{d}x \mathrm{e}^{-\mathrm{i}kx} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}q}{2\pi} \mathrm{e}^{\mathrm{i}qx} \hat{f}(q) = \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}q}{2\pi} \hat{f}(q) \int_{\mathbb{R}} \mathrm{d}x \mathrm{e}^{\mathrm{i}(q-k)x} = \hat{f}(k)$. However, if we promote $\Lambda \to \Lambda(x)$ in the first step then we can't swap the order of the two integrals and therefore we can't invert the transformation.

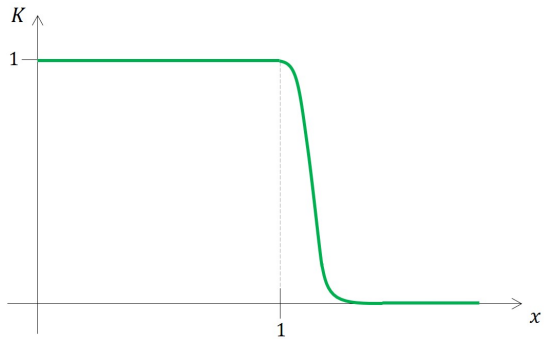


Fig. 3.1 Example of an appropriate $K(x)$

e.g. $K(x) = \mathrm{e}^{-x^2}$. In momentum space, this distinction is not necessary, as there is no issue with working with smooth but non-analytic functions. However, when we go to position space, we need to phrase these functions in terms of operators, and therefore we need them to be analytic in order to be able to define them. With that in mind we shall assume we are using *analytic* K, and therefore, Taylor expansions work.

The regulated action (with a global cutoff) is

$$S_{\Lambda_0} = \frac{1}{2}\int_p \frac{p^2+m^2}{K\left(\frac{p}{\Lambda_0}\right)}\phi(p)\phi(-p) + S_{\text{int}}[\phi]\,, \tag{3.21}$$

where, for brevity, we defined $\int_p \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d}$.

Now, using the identity [83]

$$\begin{aligned}
&\int \mathscr{D}\phi_1\, \mathscr{D}\phi_2\, \exp\left[-\frac{1}{2}\int_p \frac{1}{A(p)}\phi_1(p)\phi_2(-p) - \frac{1}{2}\int_p \frac{1}{B(p)}\phi_2(p)\phi_2(-p) + S_{\text{int}}[\phi_1+\phi_2]\right] = \\
&= \left(\int \mathscr{D}\phi\, \exp\left[-\frac{1}{2}\int_p \frac{1}{A(p)+B(p)}\phi(p)\phi(-p) - S_{\text{int}}[\phi]\right]\right) \times \\
&\left(\int \mathscr{D}\phi'\, \exp\left[\frac{1}{2}\int_p \left(\frac{1}{A(p)}+\frac{1}{B(p)}\right)\phi'(p)\phi'(-p)\right]\right)\,,
\end{aligned} \tag{3.22a}$$

where

$$\phi = \phi_1 + \phi_2 \tag{3.22b}$$

$$\phi' = -\frac{B}{A+B}\phi_1 + \frac{A}{A+B}\phi_2\,, \tag{3.22c}$$

we can write (by appropriately choosing A and B, and neglecting the ϕ' integral since it only contributes with a field independent constant)

$$\int \mathscr{D}\phi \ \mathrm{e}^{-S_{\Lambda_0}[\phi]} = \int \mathscr{D}\phi_l \mathscr{D}\phi_h \exp\left[-\frac{1}{2}\int_p \frac{p^2+m^2}{K\left(\frac{p}{\Lambda}\right)}\phi_l(p)\phi_l(-p) - \right.$$
$$\left. -\frac{1}{2}\int_p \frac{p^2+m^2}{K\left(\frac{p}{\Lambda_0}\right)-K\left(\frac{p}{\Lambda}\right)}\phi_h(p)\phi_h(-p) - S_{\mathrm{int}}[\phi_h+\phi_l]\right], \tag{3.22d}$$

which gives the required split into high and low energy modes, but now through a smooth regulator.

The key point is that, when we are integrating over the high energy modes, the propagator can be approximated via,

$$\frac{K\left(\frac{p}{\Lambda_0}\right)-K\left(\frac{p}{\Lambda}\right)}{p^2+m^2} = -\frac{pK'\left(\frac{p}{\Lambda_0}\right)\delta\Lambda}{\Lambda_0^2}\frac{1}{p^2+m^2} + \mathscr{O}(\delta\Lambda^2) \tag{3.23}$$

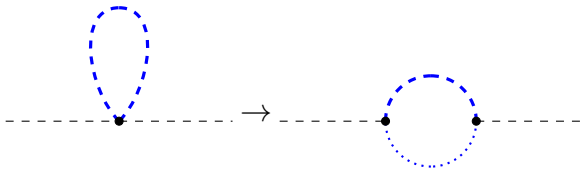
for $\Lambda = \Lambda_0 - \delta\Lambda$.

This means that, if we are only interested in the beta functions, we only need to consider diagrams with one high energy propagator. Working with a smoother cutoff implies we count propagators instead of loops. Even if we are not just interested in the beta function and we want the full RG, this is still a relevant phenomenon. The analyticity of K mean we can Taylor expand and compute the integrals order by order and different orders will not mix. We must count propagators.

This is manifestly at odds with supersymmetry. Now we cannot cancel the mass term for the scalars since the two diagrams come at a different order in $\delta\Lambda$. To counter that, we could try lowering the fermionic and scalar cutoffs at a different rate so that each scalar propagator counts as two fermionic propagators, making both terms contributing to the scalar propagator appear at the same order and allowing the mass term to cancel. However, even in that case, supersymmetry is broken. The reason now being that the corrections to the other couplings come at higher orders in $\delta\Lambda$, so the only contribution to the beta function would be from the scalar wavefunction renormalisation, and there is one scalar for the cubic coupling, but four scalars in the quartic coupling. We would not have $\lambda_{(3)}^2 = \lambda_{(4)}$ and supersymmetry would be broken.

By itself this is not a very surprising result, a similar phenomenon already happens for the much simpler four-dimensional $\mathscr{N} = 1$ theory with one complex scalar and one

Weyl fermion. In this case, however, one can preserve supersymmetry, even with a smooth regulator, by using the off-shell formalism. This is accomplished by using auxiliary fields that make the supersymmetry algebra close without using the equations of motion. This was our issue previously, by introducing a regulator in the style described above, we have changed the equations of motion, which were essential in preserving supersymmetry. Then, if we regulate all quadratic terms with the same function, including the auxiliary field, which now becomes dynamical and propagating, we do not break supersymmetry. This can happen because we no longer have the quartic scalar coupling, what we do have is a cubic coupling with two scalars and one auxiliary field. This means (using dotted lines for auxiliary fields),



(3.24)

which comes at the same order as the fermionic loop.

Knowing this result for the simpler theory, could we reproduce this with BFSS? The answer turns out to be no. Our first hurdle is the fact that no off-shell formulation with this many supercharges and finitely many fields is known[4]. We can try to ameliorate our situation by using the $\mathcal{N} = 1$ superspace formulation of four-dimensional $\mathcal{N} = 4$ SYM and dimensionally reducing it down to $1D$. In this manner we would have 4 supercharges preserved off-shell. However, this is still not enough to prevent the formation of a mass term. This happens because we do not destroy every quartic coupling, just some of them, so part of the calculation that leads to the mass term would carry through with no change. To implement a smooth cutoff, which we must do to make it local, means giving up explicit supersymmetry.

3.3 Local Renormalisation Group

In this section we do the first step in performing QRG, defining how to integrate out modes with a local regulator, *i.e.* integrating modes at different speeds in each point of spacetime. To do that we first repeat the derivation done in section 3.2.3 but now in position space. We shall see that it still holds, provided there are some restrictions on the kinetic operators we use. Then we take a particular example, of a local Gaussian regulator in one dimension and prove that that regulator obeys all necessary restrictions. This provides the first explicit realisation of a local cutoff scheme which could be used in practical calculations.

[4]We thank Nick Dorey for pointing that out to us.

3.3.1 Smooth regulator in position space

Deriving (3.22a) is rather straightforward. As it stands, we just plug in the definitions (3.22b) and (3.22c) and do the resulting algebra. The reason for this simplicity is that, in momentum space, we are dealing with ordinary multiplication of functions. In position space, however, we would be dealing with operators. Which do not obey many of the nice properties we take for granted when performing algebraic manipulations.

For simplicity, we shall resort to matrix multiplication notation, where spacetime integration is denoted with a dot product. In this notation, local operators become matrices by introducing a delta function[5],

$$-\int \mathrm{d}^d x \phi(x) \nabla^2 \phi(x) = -\int \mathrm{d}^d x \mathrm{d}^d y \, \phi(x) \left[\nabla^2 \delta^{(d)}(x-y)\right] \phi(y) = \phi^{\mathrm{T}} G^{-1} \phi \tag{3.25}$$

where

$$G^{-1}(x,y) = -\nabla^2 \delta^{(d)}(x-y) \tag{3.26}$$

and, as usual, the inverse of the operator will be it's Green's function. For example

$$G(x,y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} \mathrm{e}^{\mathrm{i}k\cdot(x-y)}\,, \tag{3.27}$$

so that,

$$G^{-1}G = \int \mathrm{d}^d z \, G^{-1}(x,z) G(z,y) = -\nabla_x^2 G(x,y) = \delta^{(d)}(x-y) = \mathbb{1}\,. \tag{3.28}$$

It is important to note that, in general, these objects will not obey the same nice properties that matrices do. Namely, for a given "matrix" (*i.e.* function of two arguments), left and right inverses do not necessarily match, and the inverse of a diagonal object is not necessarily diagonal. Note, for instance, that (3.26) is diagonal, but (3.27) is not. In this case left and right inverses do match because both are symmetric. In the end these subtleties will not be all that relevant, but it is important to have in mind the full picture.

Let us start by deriving (3.22a) in position space. We take $B^{-1} = G_\Lambda^{-1}$ to be the low energy kinetic operator and $(A+B)^{-1} = G_{\Lambda_0}^{-1}$ to be the high energy kinetic operator. We make no assumption at this point as to whether they are local or global regulators. However, by construction they will both be symmetric. So, if the inverses exist, they will behave as expected.

If we can find those two inverses, B and $A+B$, we can define $A = A+B-B = G_{\Lambda_0} - G_\Lambda$ as the high energy propagator, which is the most useful quantity in practical calculations, and,

[5]Throughout this chapter we always work in Euclidean signature.

by construction, is also symmetric. Then, if A^{-1} exists, it behaves just like a matrix inverse. Just note that finding A^{-1} can be incredibly hard because it is the opposite question to what is usually done, we have the Green's function and we want to find out the corresponding operator. However, even though our derivation will only work if such an operator actually exists, we do not actually need for any practical calculations so it suffices to show that it exists.

We repeat the derivation of (3.22a) assuming all those inverses behave as expected, and, in the next section, we present an explicit example and check whether these assumptions are valid. We will be careful in saying exactly what conditions are needed, so that, in future work it is clear if any generalisation is possible.

Analogously to (3.22b) and (3.22c), we start by defining:

$$\phi_h = A(A+B)^{-1}\phi - \phi' \tag{3.29a}$$

$$\phi_l = B(A+B)^{-1}\phi + \phi' , \tag{3.29b}$$

so that the Jacobian is still unity.

We therefore have (ignoring an overall, unimportant, factor of $\frac{1}{2}$):

$$\begin{aligned}
&\phi_h^{\mathrm{T}} A^{-1}\phi_h + \phi_l^{\mathrm{T}} B^{-1}\phi_l = \\
=&\phi^{\mathrm{T}}\left[(A+B)^{-1}\right]^{\mathrm{T}} A^{\mathrm{T}}A^{-1}A(A+B)^{-1}\phi + \phi'^{\mathrm{T}}A^{-1}\phi' - \\
-&\phi^{\mathrm{T}}\left[(A+B)^{-1}\right]^{\mathrm{T}} A^{\mathrm{T}}A^{-1}\phi' - \phi'^{\mathrm{T}}A^{-1}A(A+B)^{-1}\phi + \\
+&\phi^{\mathrm{T}}\left[(A+B)^{-1}\right]^{\mathrm{T}} B^{\mathrm{T}}B^{-1}B(A+B)^{-1}\phi + \phi'^{\mathrm{T}}B^{-1}\phi' + \\
+&\phi^{\mathrm{T}}\left[(A+B)^{-1}\right]^{\mathrm{T}} B^{\mathrm{T}}B^{-1}\phi' + \phi'^{\mathrm{T}}B^{-1}B(A+B)^{-1}\phi \,.
\end{aligned} \tag{3.30}$$

Using, $B^{\mathrm{T}}B^{-1} = \mathbb{1}$, $A^{-1}A = \mathbb{1}$, $\left[(A+B)^{-1}\right]^{\mathrm{T}}(A^{\mathrm{T}}+B^{\mathrm{T}}) = \mathbb{1}$, and $A^{\mathrm{T}}A^{-1} = \mathbb{1}$ we find,

$$\phi^{\mathrm{T}}(A+B)^{-1}\phi + \phi'^{\mathrm{T}}(A^{-1}+B^{-1})\phi' \tag{3.31}$$

which is the desired expression.

Note that if all operators and Green's functions are symmetric, which implies left and right inverses match, then all conditions are satisfied. Also note that we are free to choose both the high energy and the low energy propagators, so the real crux is on the properties of A and the existence of A^{-1}.

3.3.2 An example: local Gaussian regulator

We shall restrict to Euclidean time and consider a Gaussian regulator. In the end we shall be most interested in the case $d = 1$ but the results of this section are valid for arbitrary d. With a usual, spacetime independent cutoff we have,

$$G_\Lambda^{-1}(x_1,x_2) = -\mathrm{e}^{-\frac{\nabla_{x_2}^2}{\Lambda^2}}\nabla_{x_2}^2\delta^{(d)}(x_1-x_2) \tag{3.32a}$$

which has the Green's function,

$$G_\Lambda(x_1,x_2) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{e}^{-\frac{k^2}{\Lambda^2}}}{k^2}\mathrm{e}^{\mathrm{i}k\cdot(x_1-x_2)} \tag{3.32b}$$

such that the kinetic term looks like

$$-\int \mathrm{d}^d x\,\phi(x)\mathrm{e}^{-\frac{\nabla^2}{\Lambda^2}}\nabla^2\phi(x)\,. \tag{3.32c}$$

In what follows we need to give the cutoff spacetime dependence. If we naively just promote $\Lambda \to \Lambda(x)$ directly in (3.32c) there will be ordering issues when expanding the exponential which will make it hard to deal with. To help with that, we start with (3.32a) instead and promote $\Lambda \to \Lambda(x_1)$. In this way the derivatives actually commute with the cutoff, so there are no ordering issues. However, then the resulting operator is not symmetric (and only the symmetric part contributes to the action because it's multiplied on both sides by the same field). Therefore, we take the symmetric part and define the local version as,

$$G_\Lambda^{-1}(x_1,x_2) = -\frac{1}{2}\left(\mathrm{e}^{-\frac{\nabla_{x_2}^2}{\Lambda(x_1)^2}}\nabla_{x_2}^2 + \mathrm{e}^{-\frac{\nabla_{x_1}^2}{\Lambda(x_2)^2}}\nabla_{x_1}^2\right)\delta^{(d)}(x_1-x_2)\,. \tag{3.33}$$

Unfortunately, for arbitrary $\Lambda(x)$ we do not know how to find the Green's function of (3.33). However, for our purposes (as will be shown in the following section), we only need to find the beta functions, *i.e.* infinitesimal flow. Therefore we approximate, defining the original high energy cutoff, Λ_0 to be constant, and taking, $\Lambda(x) = \Lambda_0\mathrm{e}^{-\alpha(x)dz}$, for α and dz positive, and $dz \ll 1$. We can then solve this perturbatively, expanding in powers of dz,

$$\begin{aligned} G_\Lambda^{-1}(x_1,x_2) = &-\mathrm{e}^{-\frac{\nabla_{x_2}^2}{\Lambda_0^2}}\nabla_{x_2}^2\delta^{(d)}(x_1-x_2)+ \\ &+\frac{dz}{\Lambda_0^2}\left(\alpha(x_1)\mathrm{e}^{-\frac{\nabla_{x_2}^2}{\Lambda_0^2}}\nabla_{x_2}^4 + \alpha(x_2)\mathrm{e}^{-\frac{\nabla_{x_1}^2}{\Lambda_0^2}}\nabla_{x_1}^4\right)\delta^{(d)}(x_1-x_2) + O(dz^2)\,. \end{aligned} \tag{3.34}$$

This all means we have chosen $G_{\Lambda_0}^{-1} = (A+B)^{-1}$, $G_{\Lambda}^{-1} = B^{-1}$, for Λ_0 constant as in (3.32a) and $\Lambda(x) = \Lambda_0 \mathrm{e}^{-\alpha(x)dz}$. We then expand,

$$G_{\Lambda}(x_1,x_2) = G_{\Lambda_0}(x_1,x_2) + dz G^{(1)}(x_1,x_2) + O(dz^2) \tag{3.35}$$

giving us $A = G_{\Lambda_0} - G_{\Lambda} = -dz\,G^{(1)}$. All we have to do now is find A and show that A^{-1} exists.

First we find A, *i.e.* the Green's function for (3.34), order by order in powers of dz. At 0^{th} order, the equation is solved by construction. At 1^{st} order we get,

$$-\mathrm{e}^{-\frac{\nabla^2_{x_1}}{\Lambda_0^2}} \nabla^2_{x_1} G^{(1)}(x_1,x_2) = -\frac{\alpha(x_1)}{\Lambda_0^2} \mathrm{e}^{-\frac{\nabla^2_{x_1}}{\Lambda_0^2}} \nabla^4_{x_1} G_{\Lambda_0}(x_1,x_2) - \frac{1}{\Lambda_0^2} \mathrm{e}^{-\frac{\nabla^2_{x_1}}{\Lambda_0^2}} \nabla^4_{x_1} \big(\alpha(x_1) G_{\Lambda_0}(x_1,x_2)\big) \tag{3.36}$$

Using the definition of G_{Λ_0} as the Green's function for (3.32a) allows us to simplify the first term on the RHS,

$$-\mathrm{e}^{-\frac{\nabla^2_{x_1}}{\Lambda_0^2}} \nabla^2_{x_1} G^{(1)}(x_1,x_2) = \frac{\alpha(x_1)}{\Lambda_0^2} \nabla^2_{x_1} \delta^{(d)}(x_1-x_2) - \frac{1}{\Lambda_0^2} \mathrm{e}^{-\frac{\nabla^2_{x_1}}{\Lambda_0^2}} \nabla^4_{x_1} \big(\alpha(x_1) G_{\Lambda_0}(x_1,x_2)\big) \tag{3.37}$$

Acting with G_{Λ_0} on the left on both sides of this equation, and, once more using its defining property as the Green's function gives us,

$$G^{(1)}(x_1,x_2) = \frac{1}{\Lambda_0^2} \big(\nabla^2_{x_2} \big(G_{\Lambda_0}(x_1,x_2)\alpha(x_2)\big) + \nabla^2_{x_1} \big(\alpha(x_1) G_{\Lambda_0}(x_1,x_2)\big)\big)\,. \tag{3.38}$$

Everything is nice and symmetric as expected, which means left and right inverses will match nicely, if they do exist, that is. As mentioned above we do not actually need to find an explicit expression for the inverse, we just need to know that it indeed exists to render our calculations consistent.

It is instructive to take the Fourier transform of (3.38), using the explicit expression in (3.32b). After a straightforward calculation, just using the definition of Fourier transform and some manipulation of delta-functions we arrive at,

$$\hat{G}^{(1)}(k_1,k_2) = -\frac{1}{\Lambda_0^2} \left(\frac{k_2^2}{k_1^2} \mathrm{e}^{-\frac{k_1^2}{\Lambda_0^2}} + \frac{k_1^2}{k_2^2} \mathrm{e}^{-\frac{k_2^2}{\Lambda_0^2}} \right) \hat{\alpha}(k_1+k_2)\,. \tag{3.39}$$

Because everything is nice and symmetric, left and right inverses match, and we can then use standard linear algebra results. In this language, an inverse exists if and only if

$$\int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \hat{G}^{(1)}(k_1,k_2) f(-k_2) = 0 \ \ \forall k_1 \Rightarrow f(k_2) = 0 \ \ \forall k_2 \,, \tag{3.40}$$

where, crucially, f cannot have any dependence on k_1.

Imagine for a moment that in (3.39) there is no $\hat{\alpha}$, then this is clearly not true. We just need to pick f to be an odd function and the integral vanishes. This is also the case for constant $\hat{\alpha}$, however, a constant $\hat{\alpha}$ corresponds to a delta-function in position space, which we can clearly rule out as an allowed profile for $\hat{\alpha}$, it would correspond to changing the scale only at one point. So let us restrict to the case when $\hat{\alpha}$ is not constant.

In this case for a given $\hat{\alpha}$, and a given k_1, we could conceivably make the integral vanish for a non-zero f by judiciously choosing f, possibly relying on some non-trivial symmetry. However, because $\hat{\alpha}$ only depends on the combined sum $k_1 + k_2$, any such choice will inevitably depend on k_1. Unless $\hat{\alpha}$ is constant (which we have ruled out), by just choosing a different k_1 we will shift the profile of $\hat{\alpha}$ in an arbitrary fashion, and inevitably, some of those shifts will ruin our choice of f. Given that f cannot depend on k_1 and the condition must be valid for all k_1 we conclude that (3.40) is true, and therefore, $G^{(1)}$ is invertible, rendering our procedure consistent. We have successfully developed an RG scheme with a local change of scale.

3.4 Quantum Renormalisation Group

After developing a framework to perform local RG we can move on to the main objective of this chapter, testing quantum RG (QRG). We start by an overview of the procedure itself in greater detail than what was given in the introduction. We then move on to an overview of what is known (and is relevant) about the holographic duality of the BFSS model, to understand what should be our starting point and are we expecting to reproduce (or fail to reproduce) after performing QRG. Finally we put everything together and do the actual computation.

3.4.1 Overview of QRG

The starting point for QRG [41, 59, 60] is a quantum field theory with dynamical fields Φ which are matrix valued. These could have any spin, but it is important that they are matrix valued. We write the partition function of this theory as

$$Z = \int \mathscr{D}\Phi \ \exp(\mathrm{i}S_0[\Phi]) \tag{3.41}$$

The algorithm of QRG is as follows:

1. Turn on single trace operator deformations. In general, we should turn on a complete basis of single trace operators. However, in practice, we will only be able to turn on a finite number of them. Let O_m be the operators and $j^{(0)m}$ be the corresponding sources, the partition function is then

$$Z[j^{(0)}] = \int \mathscr{D}\Phi \ \exp\Big(\mathrm{i}S_0[\Phi] + \mathrm{i}S_1[O, j^{(0)}]\Big) \tag{3.42a}$$

where

$$S_1[O, j^{(0)}] = N^2 \sum_m \int \mathrm{d}^d x \, O_m j^{(0)m} . \tag{3.42b}$$

2. Perform an infinitesimal local change of scale, *i.e.* if in the initial theory the cutoff is Λ_0, we do an RG flow such that the new scale is $\Lambda = \mathrm{e}^{-\alpha^{(1)}(x)dz}\Lambda_0$, for $dz \ll 1$. The new partition function is, to leading order in dz,

$$Z[j^{(0)}] = \int \mathscr{D}\Phi \ \exp\Big(\mathrm{i}S_0[\Phi] + \mathrm{i}\,\delta S\,[O, j^{(0)}] + \mathrm{i}S_1[O, j^{(0)}]\Big) \tag{3.43a}$$

where

$$\begin{aligned}\delta S[O, j^{(0)}] = dz N^2 \int \mathrm{d}^d x \Big\{ & \mathscr{L}_{\mathrm{C}}(x; j^{(0)}] - \beta^m(x; j^{(0)}] O_m + \\ & + \frac{1}{2} G^{mn\{\mu\}}(x; j^{(0)}] O_m \partial_{\{\mu\}} O_n \Big\}\end{aligned} \tag{3.43b}$$

and $f(x; j^{(0)}]$ denotes a function that depends on $j^{(0)}(x)$ and its derivatives at a point x. We have used the fact we turned on a complete basis of operators to write all appearances of the fields in terms of the operators we have turned on. If we only turn on a finite number of them, it cannot generate any new ones, or otherwise this is not a consistent algorithm. Note that, to leading order in dz, we do not generate more than double trace operators.

3. Substitute $O_m \to -\frac{\mathrm{i}}{N^2}\frac{\delta}{\delta j^{(0)m}}$ in δS, noting that we must now be careful with the ordering in (3.43b), the order shown, where all the operators are on the right is the correct one

$$Z[j^{(0)}] = \int \mathscr{D}\Phi \, \exp\left(\mathrm{i}S_0[\Phi] + \mathrm{i}\,\delta S\left[-\frac{\mathrm{i}}{N^2}\frac{\delta}{\delta j^{(0)}}, j^{(0)} \right] + \mathrm{i}S_1[O, j^{(0)}] \right). \tag{3.44}$$

4. Add auxiliary fields $p_m^{(1)}$ and $j^{(1)m}$ such that

$$Z[j^{(0)}] = \int \mathscr{D}\Phi \prod_n \mathscr{D}p_n^{(1)} \mathscr{D}j^{(1)n} \, \exp\left(\mathrm{i}N^2 \int \mathrm{d}^d x \sum_m p_m^{(1)} (j^{(1)m} - j^{(0)m}) + \right.$$
$$\left. + \mathrm{i}S_0[\Phi] + \mathrm{i}\,\delta S\left[-\frac{\mathrm{i}}{N^2}\frac{\delta}{\delta j^{(1)}}, j^{(0)} \right] + \mathrm{i}S_1[O, j^{(1)}] \right). \tag{3.45}$$

5. Integrate by parts with respect to $j^{(1)m}$ in the δS term

$$Z[j^{(0)}] = \int \prod_n \mathscr{D}p_n^{(1)} \mathscr{D}j^{(1)n} \, \exp\left(\mathrm{i}N^2 \int \mathrm{d}^d x \sum_m p_m^{(1)} (j^{(1)m} - j^{(0)m}) + \right.$$
$$\left. + \mathrm{i}\,\delta S\left[-p^{(1)}, j^{(0)} \right] \right) Z[j^{(1)}]. \tag{3.46}$$

6. Now we can start with $Z[j^{(1)}]$ and iterate this procedure

$$Z[j^{(0)}] = \int \prod_{l=1}^{L} \prod_n \mathscr{D}p_n^{(l)} \mathscr{D}j^{(l)n} \, \exp\left(\mathrm{i}N^2 dz \sum_{l=1}^{L} \int \mathrm{d}^d x \sum_m p_m^{(1)} \left(\frac{j^{(l)m} - j^{(l-1)m}}{dz} \right) + \right.$$
$$\left. + \mathrm{i} \sum_{l=1}^{L} \delta S\left[-p^{(l)}, j^{(l-1)} \right] \right) Z[j^{(l)}]. \tag{3.47}$$

Taking the $dz \to 0$ limit, it's not hard to see we have generated an action that lives in $d+1$ dimensions for the new dynamical fields $j^m(z,x)$ and $p^m(z,x)$. It is important to note that if no double trace operators are generated then this action will be linear in $p^m(z,x)$ and therefore this field will still just be a Lagrange multiplier, not a dynamical field. In order to have non-trivial dynamics for these fields we must generate double trace operators.

The conjecture is that Gauge/Gravity Duality is completely encapsulated in a procedure such as this one. As mentioned in the introduction, there has been some additional work with

regards to this conjecture. Namely some hints for it's application to the original AdS_5/CFT_4 case [71], a concrete calculation for the $U(N)$ vector model [72], and, understanding the conditions under which one can recover full $(d+1)$-dim diffeomorphism invariance [73]. This last one is the most relevant for our purposes since it his here that the importance of having a spacetime dependent cutoff was fully appreciated as a means to recover diffeomorphism invariance.

3.4.2 Overview of the holographic dual to BFSS

As mentioned in section 3.2.1, the BFSS model, describes the dynamics of N coincident D0-branes. This means it also has a dual gravitational description in terms of 10-dim type IIA supergravity [75]. In the decoupling limit,

$$U = \frac{r}{\alpha'} = \text{fixed},\ g^2_{\text{YM}} = \frac{1}{4\pi^2}\frac{g_s}{\alpha'^{3/2}} = \text{fixed},\ \alpha' \to 0 \tag{3.48}$$

the supergravity background solution corresponding to BFSS is given by [75],

$$\mathrm{d}s^2 = \alpha'\left(-\frac{U^{7/2}}{4\pi^2 g_{\text{YM}}\sqrt{15\pi N}}\mathrm{d}t^2 + \frac{4\pi^2 g_{\text{YM}}\sqrt{15\pi N}}{U^{7/2}}\mathrm{d}U^2 + \frac{4\pi^2 g_{\text{YM}}\sqrt{15\pi N}}{U^{3/2}}\mathrm{d}\Omega^2\right), \tag{3.49a}$$

$$\mathrm{e}^{\phi} = 4\pi^2 g^2_{\text{YM}}\left(\frac{240\pi^5 g^2_{YM} N}{U^7}\right)^{3/4}. \tag{3.49b}$$

where $\mathrm{d}\Omega^2$ is the metric on a round unit radius S^8, α' is related to the string length and g_s is the string coupling. We note in passing that strictly speaking this solution is singular at the origin. The standard way to deal with this is to put the system at a finite temperature, which corresponds to having a black hole in the gravity perspective. However, if we are far enough away from the origin, *i.e.* near the boundary, the effects of this temperature should be minimal, that is also the region we are we have more control over our field theoretic description. Therefore, in this chapter, we neglect finite temperature effects.

Another point to make is that, as mentioned in [75], the curvature gets large as we approach the boundary, more specifically, $\alpha' R \sim \sqrt{\frac{U^3}{g^2_{\text{YM}}N}}$, and therefore we have less faith on our supergravity description. Naively, this does not intersect with the region where we have analytic control on the field theory side. However, in QRG we only need to do one infinitesimal step of coarse graining, and, as we have showcased in sections 3.2.3 and 3.3.2 we can do that exactly. This seems to solve all our problems, but there is an issue. The

theory we want is the one that approaches the action (3.2) in the UV. When the coupling is strong, the correct action is not (3.2) but it needs corrections that, by construction, will be very important. If we just take the action (3.2) and define the coupling to be strong we have a well defined theory and calculations, it will simply not be the theory we're looking after. This is similar to how we can solve QCD in the strong coupling limit exactly using lattice methods but the answers we get aren't physically relevant[6].

The resolution to this issue comes from the realisation that, in the gravity side, we should insert the sources at the boundary, not deep into the bulk. Therefore, we should start with a field theory action in the $\Lambda_0 \to \infty$ limit. Then we do the one infinitesimal coarse graining step required by QRG at this weak coupling limit. By the nature of QRG we can put all corrections due to this step into the new dynamical fields and start again with the original action. This means we can confidently do all the hard calculations in the regime where we have control over the theory and then use the auxiliary turned dynamical fields to recover the important physics.

With those points in mind we carry on with our discussion. The solution presented above is not the full content of the gauge/gravity duality. As was mentioned in the introduction, the most general form of the correspondence is an equality between partition functions that allows us to calculate correlation functions on both sides (and hopefully match them) [25–28]. However, to do that, we need to find out which operators on the field theory side correspond to which modes on the gravity side.

This is precisely what was done in [76]. By decomposing the ten-dimensional modes in harmonics of the eight-dimensional sphere, they have found a correspondence between certain supergravity modes and certain operators discussed in [89]. In addition to harmonic analysis, a very important tool is generalised conformal symmetry, which, despite its importance, is not very pertinent to the main point of this chapter, so we skip it, for interested readers here is a selection of useful literature on the subject [45, 90–94].

We will not repeat here the full dictionary except to point out that these modes are constructed such that, up to quadratic order in the supergravity action, they do not mix and have an effective two-dimensional action. Therefore, if we turn on the correct operator in the field theory side, even if just that one, we should be capable of reproducing the correct 2-point function on the gravity side. This test has indeed been made in [80] and matching between the two sides has been found.

In particular we shall turn on the operator [89]

$$T^{++}_{2,ij} = \frac{1}{\lambda^{9/7} N} \mathrm{Tr}\left(X_i X_j - \frac{\delta_{ij}}{9} X_k X_k \right) \tag{3.50}$$

[6] We thank David Tong for pointing this out.

which is dual to the supergravity mode [76]

$$s_3^{\ell=2} = z^{-7/5}\left(-7b_i^i + f\right) \tag{3.51}$$

where

$$z = \frac{2}{5}q^{1/2}r^{-5/2} \tag{3.52a}$$
$$q = 60\pi^3(\alpha')^{7/2}g_sN \tag{3.52b}$$
$$f = g_s\left(\frac{5}{2}\right)^{19/5} q^{-2/5}z^{19/5}(\partial_0 a_z - \partial_z a_0) \tag{3.52c}$$
$$h_i^i(x^\mu) = \sum b_i^i(t,z)Y(x^i) \tag{3.52d}$$
$$\hat{A}_z(x^\mu) = \sum a_z(t,z)Y(x^i) \tag{3.52e}$$
$$\hat{A}_0(x^\mu) = \sum a_0(t,z)Y(x^i)\,, \tag{3.52f}$$

and Y are the scalar $SO(9)$ spherical harmonics (we have suppressed their internal indices), $h_{\mu\nu}$, $\hat{A}_\mu$ are the perturbations of, respectively, the metric and the gauge field around the background (3.49a).

These modes, have the following 2-point function, as discussed in [76] and confirmed in [80]:

$$\left\langle \mathscr{O}(\tau)\mathscr{O}(\tau')\right\rangle_c = \frac{1}{\kappa^2}q^{29/35}\frac{1}{|\tau-\tau'|^{-1/5}} \tag{3.53}$$

which we should be able to reproduce if QRG is valid. We shall assess in the following whether or not QRG holds.

Finally, a quick note that, for this simple case, there is the possibility of recovering interactions because there are known fully consistent truncations down to 2 dimensions [77–79], which do agree with the tests performed in [80]. Even though we shall use the fact such truncations exist to draw some conclusions we shall not need to use the particular structure therefore we refer the reader to the above cited literature.

3.4.3 QRG of BFSS

We now apply the full QRG calculation of the action (3.2). We first add the source term,

$$N^2\int \mathrm{d}\tau J_{2,ij}^{++}(\tau)T_{2,ij}^{++}(\tau) \tag{3.54}$$

where $T_{2,ij}^{++}$ is given by (3.50).

In this case, because it is very important to keep track of the trace structure we shall resort to fundamental indices $I,J = 1,\ldots,N$, and represent the fields by traceless Hermitian matrices. It is also simpler to use Wick's theorem instead of Feynman diagrams. We shall furthermore be agnostic about the regulator procedure used, just noting that is has to be local, and could be, for example, the one developed in section 3.3.2 (it is not too hard to generalise the results of that section to include fermions). We then split all the index structure apart from the temporal dependence and write, for the high energy modes to be integrated out,

$$\left\langle \tilde{X}_i^{+I}{}_J(\tau_1)\tilde{X}_j^{+K}{}_L(\tau_2)\right\rangle_+ = A_X(\tau_1,\tau_2)\delta_{ij}\left(\delta_L^I\delta_J^K - \frac{1}{N}\delta_J^I\delta_L^K\right) \tag{3.55a}$$

$$\left\langle \tilde{\psi}_{\alpha J}^{+I}(\tau_1)\tilde{\psi}_{\beta L}^{+K}(\tau_2)\right\rangle_+ = A_\psi(\tau_1,\tau_2)\delta_{\alpha\beta}\left(\delta_L^I\delta_J^K - \frac{1}{N}\delta_J^I\delta_L^K\right) \tag{3.55b}$$

where we are using the rescaled variables defined in (3.4).

All we have to do is compute all connected correlation functions with just a single contraction, *i.e.* propagator. Up to that order, the only terms that contribute are those that come from the expectation value of a single operator, or from the expectation value of the product of two operators. All such calculations proceed in exactly the same manner: expand the expectation value; pick all possible pairs of fields to be "+", *i.e.* high energy, summing over all possible choices, the remaining fields become "-"; (anti-)commute past each other (depending if they're scalars or fermions) until you have expressions of the form (3.55a) or (3.55b); contract all indices noting that $\delta_I^I = N$. Therefore, we shall only present the full details for the first calculation and for all others we merely give the final answer. We note, however, that everything that involves the quartic interaction is much more cumbursome than anything else, because we need to sum over the possible choices.

Single operator:

Cubic interaction:

$$\begin{aligned}
&\int \mathrm{d}\tau \left\langle \frac{1}{2}\sqrt{\frac{\lambda}{N}}\,\mathrm{Tr}\left\{\tilde{\psi}_\alpha(\gamma_i)_{\alpha\beta}\left[\tilde{X}_i,\tilde{\psi}_\beta\right]\right\}\right\rangle_+ = \\
&= \int \mathrm{d}\tau\,(\gamma_i)_{\alpha\beta}\frac{1}{2}\sqrt{\frac{\lambda}{N}}\,\mathrm{Tr}\left\{\left\langle \tilde{\psi}_\alpha^+\left[\tilde{X}_i^-,\tilde{\psi}_\beta^+\right]\right\rangle\right\}_+ = \\
&= \int \mathrm{d}\tau\,(\gamma_i)_{\alpha\beta}\frac{1}{2}\sqrt{\frac{\lambda}{N}}\left(\left\langle \tilde{\psi}_{\beta J}^{+I}\tilde{\psi}_{\alpha K}^{+J}\right\rangle_+ \tilde{X}_{i\,I}^{-K} - \left\langle \tilde{\psi}_{\alpha J}^{+I}\tilde{\psi}_{\beta K}^{+J}\right\rangle_+ \tilde{X}_{i\,I}^{-K}\right) = \\
&= \int \mathrm{d}\tau\,(\gamma_i)_{\alpha\beta}\frac{1}{2}\sqrt{\frac{\lambda}{N}}A_\psi(\tau,\tau)\delta_{\alpha\beta}\left(\left(\delta_K^I N - \frac{1}{N}\delta_K^I\right) - \left(\delta_K^I N - \frac{1}{N}\delta_K^I\right)\right)\tilde{X}_{i\,I}^{-K} = 0
\end{aligned} \tag{3.56}$$

Quartic interaction:

$$\int \mathrm{d}\tau \left\langle -\frac{\lambda}{4N}\mathrm{Tr}\Big\{\big[\tilde{X}_i,\tilde{X}_j\big]^2\Big\}\right\rangle_+ = 8\lambda\int \mathrm{d}\tau A_X(\tau,\tau)\,\mathrm{Tr}\big\{(\tilde{X}_i^-)^2\big\} \tag{3.57}$$

Source term:

$$N^2\int \mathrm{d}\tau J^{++}_{2,ij}(\tau)\left\langle T^{++}_{2,ij}(\tau)\right\rangle_+ = 0 \tag{3.58}$$

Two operators

Cubic-Cubic:

$$\begin{aligned}\Big\langle \int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\frac{\lambda}{4N}\mathrm{Tr}\big\{\tilde{\psi}_\alpha(\tau_1)(\gamma_i)_{\alpha\beta}\big[\tilde{X}_i(\tau_1),\tilde{\psi}_\beta(\tau_1)\big]\big\}\,\mathrm{Tr}\big\{\tilde{\psi}_\gamma(\tau_2)(\gamma_j)_{\gamma\delta}\big[\tilde{X}_j(\tau_2),\tilde{\psi}_\delta(\tau_2)\big]\big\}\Big\rangle_{+,c} =\\ =-\frac{\lambda}{N}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_\psi(\tau_1,\tau_2)\,\mathrm{Tr}\Big\{\Big[\tilde{X}_i^-(\tau_1),(\gamma_i)_{\alpha\beta}\tilde{\psi}^-_\beta(\tau_1)\Big]\Big[\tilde{X}_j^-(\tau_2),(\gamma_j)_{\alpha\gamma}\tilde{\psi}^-_\gamma(\tau_2)\Big]\Big\}+\\ +\frac{\lambda}{4N}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_X(\tau_1,\tau_2)\,\mathrm{Tr}\Big\{(\gamma_i)_{\alpha\beta}\Big\{\tilde{\psi}^-_\alpha(\tau_1),\tilde{\psi}^-_\beta(\tau_2)\Big\}(\gamma_i)_{\gamma\delta}\Big\{\tilde{\psi}^-_\gamma(\tau_1),\tilde{\psi}^-_\delta(\tau_2)\Big\}\Big\}\end{aligned} \tag{3.59}$$

Cubic-Quartic:

$$\begin{aligned}\Big\langle -\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\left(\frac{\lambda}{4N}\right)^{3/2}\mathrm{Tr}\big\{\tilde{\psi}_\alpha(\tau_1)(\gamma_i)_{\alpha\beta}\big[\tilde{X}_i(\tau_1),\tilde{\psi}_\beta(\tau_1)\big]\big\}\,\mathrm{Tr}\Big\{\big[\tilde{X}_j(\tau_2),\tilde{X}_k(\tau_2)\big]^2\Big\}\Big\rangle_{+,c} =\\ =\frac{1}{2}\left(\frac{\lambda}{N}\right)^{3/2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,(\gamma_i)_{\alpha\beta}A_X(\tau_1,\tau_2)\,\mathrm{Tr}\Big\{\Big\{\tilde{\psi}^-_\alpha(\tau_1),\tilde{\psi}^-_\beta(\tau_1)\Big\}\big[\tilde{X}_k^-(\tau_2),\big[\tilde{X}_i^-(\tau_2),\tilde{X}_k^-(\tau_2)\big]\big]\Big\}\end{aligned} \tag{3.60}$$

Cubic-Source:

$$\begin{aligned}\Big\langle -\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\sqrt{\frac{\lambda}{4N}}\mathrm{Tr}\big\{\tilde{\psi}_\alpha(\tau_1)(\gamma_i)_{\alpha\beta}\big[\tilde{X}_i(\tau_1),\tilde{\psi}_\beta(\tau_1)\big]\big\}N^2J^{++}_{2,jk}(\tau_2)T^{++}_{2,jk}(\tau_2)\Big\rangle_{+,c} =\\ =-\sqrt{\frac{\lambda}{N}}\frac{1}{\lambda^{2/7}}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_X(\tau_1,\tau_2)\Big[(\gamma_i)_{\alpha\beta}J^{++}_{2,(ij)}(\tau_2)\,\mathrm{Tr}\Big\{\Big\{\tilde{\psi}^-_\alpha(\tau_1),\tilde{\psi}^-_\beta(\tau_1)\Big\}\tilde{X}_j^-(\tau_2)\Big\}-\\ -\frac{1}{9}(\gamma_i)_{\alpha\beta}J^{++}_{2,jj}(\tau_2)\,\mathrm{Tr}\Big\{\Big\{\tilde{\psi}^-_\alpha(\tau_1),\tilde{\psi}^-_\beta(\tau_1)\Big\}\tilde{X}_i^-(\tau_2\Big\}\Big]\end{aligned} \tag{3.61}$$

Quartic-Quartic:

$$\left\langle \frac{\lambda^2}{16N^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{Tr}\Big\{\big[\tilde{X}_i(\tau_1),\tilde{X}_j(\tau_1)\big]^2\Big\}\,\mathrm{Tr}\Big\{\big[\tilde{X}_k(\tau_2),\tilde{X}_l(\tau_2)\big]^2\Big\}\right\rangle_{+,c} = \\ = \frac{\lambda^2}{N^2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_X(\tau_1,\tau_2)\,\mathrm{Tr}\Big\{\Big[\tilde{X}_i^-(\tau_1),\Big[\tilde{X}_j^-(\tau_1),\tilde{X}_i^-(\tau_1)\Big]\Big]\cdot \\ \cdot\Big[\tilde{X}_k^-(\tau_2),\Big[\tilde{X}_j^-(\tau_2),\tilde{X}_k^-(\tau_2)\Big]\Big]\Big\} \tag{3.62}$$

Quartic-Source:

$$\left\langle -\frac{\lambda}{4N}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{Tr}\Big\{\big[\tilde{X}_i(\tau_1),\tilde{X}_j(\tau_1)\big]^2\Big\}N^2J^{++}_{2,jk}(\tau_2)T^{++}_{2,jk}(\tau_2)\right\rangle_{+,c} = \\ = -\frac{\lambda^{5/7}}{N}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_X(\tau_1,\tau_2)J^{++}_{2,kl}(\tau_2)\,\mathrm{Tr}\Big\{\big[\tilde{X}_i^-(\tau_1),\tilde{X}_k^-(\tau_1)\big]\big[\tilde{X}_i^-(\tau_1),\tilde{X}_l^-(\tau_2)\big]- \\ -\frac{\delta_{kl}}{9}\Big[\tilde{X}_i^-(\tau_1),\tilde{X}_j^-(\tau_1)\Big]\Big[\tilde{X}_i^-(\tau_1),\tilde{X}_j^-(\tau_2)\Big]\Big\} \tag{3.63}$$

Source-Source:

$$\left\langle \int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 N^2J^{++}_{2,ij}(\tau_2)T^{++}_{2,ij}(\tau_1)N^2J^{++}_{2,kl}(\tau_2)T^{++}_{2,kl}(\tau_2)\right\rangle_{+,c} = \\ = \frac{4}{\lambda^{4/7}}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 A_X(\tau_1,\tau_2)J^{++}_{2,ij}(\tau_1)J^{++}_{2,kl}(\tau_2)\,\mathrm{Tr}\Big\{\delta_{jk}\tilde{X}_i^-(\tau_1)\tilde{X}_l^-(\tau_2)- \\ -\frac{\delta_{kl}}{9}\tilde{X}_i^-(\tau_1)\tilde{X}_j^-(\tau_1)-\frac{\delta_{ij}}{9}\tilde{X}_k^-(\tau_1)\tilde{X}_l^-(\tau_2)+\frac{\delta_{ij}\delta_{kl}}{81}\tilde{X}_m^-(\tau_1)\tilde{X}_m^-(\tau_2)\Big\} \tag{3.64}$$

These results at first sight look quite daunting, as it does not look very clear how to interpret them. The main reason is that we have generated many new operators which were not there to begin with, and, some of them, violate supersymmetry (like the mass term). Something which we had already anticipated could happen due to the results from section 3.2.3. However, there is one simplifying aspect, there are no double trace operators.

Naively, this seems rather fatal. As we pointed out in section 3.4.1, if there are no double trace operators then there are no non-trivial dynamics for the new fields. This seems to be in stark contrast with the predictions in [76] which predicts a non-trivial 2-point function for this mode, and with [80] which checked it numerically. Note that it cannot be an artefact of us having neglected temperature since in [80] finite temperature effects are also neglected and still they find non-trivial dynamics. It also cannot be an artefact over our choice of vacuum (namely, we expanded about the trivial vacuum) since in [80] they used the same vacuum.

One could also worry that this is an artefact of the breaking of supersymmetry. However, our concern is that we have generated too few operators, if we had used a supersymmetry preserving regulator then the most that could have happened is a cancellation between separate diagrams, which would mean generating even fewer diagrams, which wouldn't solve this issue. Further, given that QRG only works if the regulator is spacetime dependent and *all* local RG schemes break supersymmetry (as discussed in sections 3.2 and 3.3), no scheme consistent with the original QRG proposal is capable of preserving supersymmetry, therefore this must be interpreted as a feature of the proposal itself.

However, there is still a possibility that we have missed something. The trick lies in the extra operators that we have generated[7]. If we don't have any reason to truncate them we should consider them in our analysis, however, the only way to do so seems to be adding sources for those operators in step 1. This by itself also goes against the supergravity predictions, this mode should have dynamics on its own, not just when coupled to other operators (and in the lattice simulations dynamics where observed without the need to turn on more operators). However, after we do this, we can take the limit where the original sources are all set to zero and then carry out the calculation anyway, possibly finding non-zero double trace operators which will only turn on away from the boundary. Then, technically, we have only turned on that single mode initially, it just so happened to turn on other modes which then gave it the necessary dynamics. This mechanism cannot be completely ruled out by our calculations, and it seems that our simplifying assumption that we only need to turn on a finite set of sources and still get meaningful answers is not justified, however, in practice, it is not possible (nor naively well defined) to turn on an infinite number of operators. This leads to many difficulties in proceeding and confirming or completely ruling out QRG, which the authors leave as open problems.

Firstly, it shouldn't be surprising that we have turned on extra modes, this is not a consistent truncation after all, this mode interacts with others. Therefore, in order to correctly interpret the results there should be some consistent way to truncate and neglect some operators to reproduce the approximation made in the supergravity side. However, neither the large N limit nor generalised conformal dimensions seem to do the trick since all single trace operators scale equally in the large N limit and in $d = 1$ the fields have negative dimensions, so having more fields will lower the dimension even further.

To deal with this, one could try to use a consistent truncation instead. However, some of the single trace operators we have generated above are not part of the consistent truncation. This is problematic unless they never become dynamical. So we still run into the issue of having to turn on an infinite number of operators, with the added fact that we know that if

[7] We thank Sung-Sik Lee for pointing this out

QRG is valid then we can only generate double trace operators for those exact operators we turned on initially, we may still need an infinite number of auxiliary non-dynamical fields. The extent to which having those fields will affect physical results is unclear.

Finally, we note that, even in the case when no source is turned on, we still generate some single trace operators. None of these modes may at any point become dynamical because that would mean that the vacuum has non-trivial dynamics, which, once more, goes against the supergravity predictions. However, this is still not a full contradiction since it may be that these new modes are never dynamical unless we turn on sources at the start.[8]

This leads us with a very narrow window of possible success for QRG, it cannot generate any non-trivial dynamics when no source is turned on, it must generate non-trivial dynamics when any of the sources in [76] is turned on, and it cannot generate non-trivial dynamics away from the consistent truncations in [77–79] when only those modes are initially turned on. Perhaps some clever use of $SO(9)$ symmetry could constrain which modes are turned on at each step and confirm or rule out QRG, however, currently the authors are unaware of any such method.

3.5 Discussion

There were three main steps in this chapter: doing global RG on the BFSS model, developing a local RG scheme, and performing QRG on BFSS. The first two were part of the necessary construction to perform QRG, but they are also very important and interesting in their own right.

First of all, we performed standard Wilsonian RG on the BFSS model. This result was absent to the literature due to the finiteness of BFSS but was a very useful warm-up calculation. Even more importantly, it highlighted under which conditions were we able to preserve supersymmetry along the flow. Namely, a hard cutoff breaks supersymmetry but if we use Feynman parameters, as is usually done in higher dimensions, supersymmetry appears to be preserved. This is very surprising and the interpretation is not yet clear, because the physical hard cutoff breaks supersymmetry, the Feynman parametrisation is a mere computational trick. Furthermore, we concluded that the use of a smooth regulator always breaks supersymmetry. Even the use of the superspace formalism does not help because it

[8] This also doesn't constrain the single-mode calculation too much, because even though the $\mathrm{Tr}(X^{2075})$ mode, for example, generated by the pure vacuum (modes of this form are eventually generated) cannot generate double-trace operators by itself or with other vacuum operators, it may still generate double trace operators when contracted with one of the modes turned on by the sources. So, even though by themselves they are non-dynamical we cannot just throw them away.

does not preserve enough supersymmetry off-shell, it only preserves 4 supercharges out of the 16 total.

Secondly, we discussed under which conditions can we use a local regulator, and constructed an explicit example of one, a local Gaussian regulator. Constructing a local regulator is harder than a global one because of the subtleties of dealing with infinite dimensional objects, but we have shown that it is possible to do it, so long as we make sure every operator is symmetric and has an inverse. This section is especially interesting because it could potentially be used for performing RG in curved spacetime.

Finally, we put all the pieces together and performed QRG on BFSS with a particular operator turned on, one which we know from independent studies that has non-trivial dynamics in the gravity side, and found that it didn't generate any double trace operators. Further considerations meant it didn't completely rule out QRG but it greatly limited the ways in which it could still work. So far it appears to require turning on an infinite set of operators which is unclear if it is possible to do in practice. But further studies are necessary to fully understand its role in understanding the AdS/CFT correspondence.

Chapter 4

Testing the AdS/CFT correspondence with stringy corrections to type IIB supergravity

In the previous chapter we explored the realm of local and quantum RG as a means to better understand holography. However, in the end, we had a null result, quantum RG does not appear to work. So perhaps it is worth delving into the evidence for AdS/CFT, do the predictions regarding the matching of the scales really work as well as we hope?

This chapter presents a test of this reasoning, by studying the leading α' corrections to the entropy of certain black holes with $AdS_5 \times S^5$ asymptotics. We find that, in the supersymmetric limit, the entropy does not receive α' corrections. This result strengthens recent calculations that match the index of $\mathcal{N} = 4$ Super-Yang-Mills with the corresponding partition function in the supersymmetric limit. In the small temperature regime, we find that the entropy corrections are concordant with the weak gravity conjecture.

With this result we have strengthened that our understanding of both Super-Yang-Mills and the AdS/CFT corresponding are accurate. Now, future research in this field has a stronger ground to stand on and explore.

Once again all the relevant models used will be introduced to the level of depth required, and the relevant details of the AdS/CFT correspondence needed to understand the arguments will be included.

This chapter is based on [2] written in collaboration with Jorge E. Santos.

4.1 Introduction

Quantum gravity remains a largely unexplored frontier. However, due to the seminal work in black hole thermodynamics in the seventies [95–103] we know that, whatever the ultimate unifying theory is, it should reproduce the Hawking effect and give a microscopic derivation of the Bekenstein-Hawking black hole entropy in the appropriate semi-classical limit. To date, string theory appears to be the only candidate for a quantum theory of gravity that explains both of these effects in an ambiguity free manner at a microscopic level [104–107]. In particular, the seminal work of [104] provided a beautiful matching between the Bekenstein-Hawking entropy of certain five-dimensional supersymmetric black holes with asymptotically flat boundary conditions and the counting of specific supersymmetric states. Since then, a number of generalisations of this work have been accomplished for black holes with more complex topologies (see *e.g.* [108]).

However, this matching has only been accomplished for black holes with asymptotically flat boundary conditions. One might wonder how to extend these results to asymptotically anti-de Sitter (AdS) spacetimes, for which we have the so-called AdS/CFT correspondence [25–28]. In its original form, the AdS/CFT correspondence relates four-dimensional $\mathscr{N}=4$ Super-Yang-Mills (SYM) with gauge group $SU(N)$ and 't Hooft coupling λ, to type IIB superstring theory with string coupling g_s, string length $\ell_s \equiv \sqrt{\alpha'}$ on $\mathrm{AdS}_5 \times S^5$ with radius L and N units of $F_{(5)}$ flux through the S^5. The field theory is thought to live at the conformal boundary of AdS_5, and for this reason the correspondence is said to be holographic in nature. The string theory side is often referred to as the 'bulk' and the field theory side as the 'boundary'.

The parameters on each side of the AdS/CFT correspondence are related via

$$\frac{\lambda}{N} = 2\pi g_s \qquad \text{and} \qquad 2\lambda = \frac{L^4}{\ell_s^4}\,. \tag{4.1}$$

However, it remains a challenge to understand string theory for generic values of g_s, so one usually takes $N \to +\infty$, at fixed λ, so that $g_s \to 0$. Under these assumptions, the bulk theory reduces to a classical theory of strings. To simplify matters further, we can also take λ to be large, but not necessarily infinite. On the field theory side, we are thus looking at strong coupling effects, and on the gravity side we have a supergravity theory. Corrections to the strict $\lambda \to +\infty$ limit appear in the bulk as higher derivative terms which account for finite size string corrections.

The problem of reproducing the entropy of certain black hole solutions in global AdS_5 on the string theory side is now mapped into a counting problem of certain states on the

field theory. Because we are interested in global AdS_5, the field theory is thought to live on $\mathbb{R}_t \times S^3$. The holographic description of electrically-charged supersymmetric black holes with $AdS \times S^5$ asymptotics is in terms of states of the dual $\mathscr{N} = 4$ SYM that preserve only one of the available sixteen supercharges. Such states should be counted (with sign) by the superconformal index. However, early attempts to compute this index gave an order one result [109], whereas the entropy of AdS_5 black holes scales with N^2. It was not until recently that this long-standing problem was partially solved. In particular, [110–126] have argued that, upon using complex chemical potentials, the cancellations between fermionic and bosonic degrees of freedom observed in [109] can be avoided. This leads to an index of order N^2, whose associated entropy matches those of known supersymmetric black holes [127–129]. This body of work thus provides overwhelming evidence that whether we compute the entropy via the index or via a more standard calculation using the partition function of $\mathscr{N} = 4$ SYM, the results should agree with each other. It should be noted that this latter quantity can only be computed via an indirect bulk calculation using the Bekenstein-Hawking entropy.

The matching between the partition function calculation and index, leads to a number of fascinating predictions. In particular, since the index cannot exhibit a dependence on continuous parameters[1], we expect the counting on the field theory side to not depend on the 't Hooft coupling λ. On the bulk side of the story, because we are computing directly a partition function, this is not an obvious fact given we know that the classical equations of motion of type IIB supergravity do admit corrections in α', due to finite size stringy effects. These, in the small α' limit, appear as higher-derivative corrections to the equations of motion of type IIB supergravity. The first non-trivial corrections for supergravity configurations that only involve the metric g and five-form $F_{(5)}$ were worked out in [3][2], following the seminal results of [132].

4.2 The black holes

We focus on black hole solutions of five-dimensional minimal gauged supergravity, whose action comprises a five-dimensional metric g and a field strength $F = \mathrm{d}A$ and reads

$$S_{5D} = \frac{1}{16\pi G_5}\int_{\mathscr{M}} \mathrm{d}^5x\sqrt{-g}\left(R + \frac{12}{L^2} - \frac{1}{4}F_{ab}F^{ab} + \frac{1}{12\sqrt{3}}\varepsilon^{abcde}F_{ab}F_{cd}A_e\right). \quad (4.2)$$

[1] Except perhaps when wall-crossing is observed, see *e.g.* [130, 131].

[2] We would like to note, however, that [3] has a number of typos in their section 4, which summarises their results.

Known black hole solutions in this theory carry one electric charge Q, and two angular momenta J_1, J_2. For simplicity, we focus on the case where $J_1 = J_2 = J$. The equations of motion derived from Eq. (4.2) read

$$R_{ab} - \frac{g_{ab}}{2}R - \frac{6}{L^2}g_{ab} = \frac{1}{2}\left(F_a{}^c F_{bc} - \frac{g_{ab}}{4}F^{cd}F_{cd}\right), \tag{4.3a}$$

$$\nabla_a F^{ab} = \frac{1}{4\sqrt{3}}\varepsilon^{bcdef}F_{cd}F_{ef}\,. \tag{4.3b}$$

We are interested in the α' corrections to the entropy of the black holes constructed in [133], which read

$$\mathrm{d}s^2_{5D} = -\frac{f}{h}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f} + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2) + \frac{r^2}{4}h\,(\sigma_3 - W\,\mathrm{d}t)^2\,, \tag{4.4a}$$

$$A = \frac{\sqrt{3}\tilde{Q}}{r^2}\left(\mathrm{d}t - \frac{\tilde{J}}{2}\sigma_3\right), \tag{4.4b}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the usual left-invariant 1-forms of S^3

$$\sigma_1 = -\sin\psi\;\mathrm{d}\theta + \cos\psi\sin\theta\;\mathrm{d}\phi\,, \tag{4.5a}$$

$$\sigma_2 = \cos\psi\;\mathrm{d}\theta + \sin\psi\sin\theta\;\mathrm{d}\phi\,, \tag{4.5b}$$

$$\sigma_3 = \mathrm{d}\psi + \cos\theta\;\mathrm{d}\phi\,, \tag{4.5c}$$

and

$$f = \frac{r^2}{L^2} + 1 - \frac{2\tilde{M}}{r^2}(1-\chi) + \frac{\tilde{Q}^2}{r^4}\left(1 - \frac{\tilde{J}^2}{L^2} + \frac{2\tilde{M}L^2\chi}{\tilde{Q}^2}\right), \tag{4.6a}$$

$$W = \frac{2\tilde{J}}{r^2 h}\left(\frac{2\tilde{M}+\tilde{Q}}{r^2} - \frac{\tilde{Q}^2}{r^4}\right), \tag{4.6b}$$

$$h = 1 - \frac{\tilde{J}^2\tilde{Q}^2}{r^6} + \frac{2\tilde{J}^2(\tilde{M}+\tilde{Q})}{r^4}, \tag{4.6c}$$

where $L^2\chi \equiv \tilde{J}^2(1+\tilde{Q}/\tilde{M})$. The constants $\tilde{M}$, $\tilde{Q}$ and $\tilde{J}$ parametrise the energy M, electric charge Q and angular momentum J as

$$M = \frac{3\tilde{M}\pi}{4G_5}\left(1 + \frac{\chi}{3}\right), \tag{4.7a}$$

$$J = \frac{\tilde{J}\pi}{4G_5}(2\tilde{M}+\tilde{Q})\,, \tag{4.7b}$$

$$Q = \frac{\sqrt{3}L\pi\tilde{Q}}{4G_5}\,. \tag{4.7c}$$

The black hole event horizon is the null hypersurface $r = r_+$, with r_+ being the largest real positive root of $f(r)$. The associated Hawking temperature T, entropy S, chemical potential μ and angular velocity Ω can be found in [133]. It is then a simple exercise to check that all thermodynamic quantities satisfy the first law of black hole mechanics

$$\mathrm{d}E = T\,\mathrm{d}S + \mu\,\mathrm{d}Q + \Omega\,\mathrm{d}J\,. \tag{4.8}$$

The Gibbs free energy is then constructed in the usual manner via $G = E - TS - \mu Q - \Omega J$. One can show that G/T agrees with the Euclidean on-shell action (4.2) up to the usual Gibbons-Hawking-York [134, 135] term and boundary counterterms [136, 43].

Finally, with our normalizations for F, the BPS condition is given by[3]

$$\Delta \equiv M - \frac{2}{L}J - \frac{\sqrt{3}}{L}Q \geq 0\,. \tag{4.9}$$

The saturation of the BPS condition occurs only for supersymmetric solutions. Similar BPS bound have been shown not to receive α' corrections even for asymptotically flat black holes [137]. The AdS BPS condition (4.9), together with the first law, implies $T = 0$, $\Omega = 2/L$ and $\mu = \sqrt{3}/L$, which in turn yield

$$\tilde{Q} = \tilde{Q}_{\mathrm{BPS}} \equiv r_+^2\left(1 + \frac{r_+^2}{2L^2}\right), \tag{4.10a}$$

$$\tilde{J} = \tilde{J}_{\mathrm{BPS}} \equiv \frac{L r_+^2}{r_+^2 + 2L^2}\,. \tag{4.10b}$$

Note that even though the solutions (4.4) appear to depend on three parameters $(\tilde{M}, \tilde{Q}, \tilde{J})$, the BPS condition reduces this family to a one-parameter family, despite the fact that extremal black holes form a two-parameter family of solutions. We remark that [138, 139] provided strong numerical evidence for the existence of a new two-parameter family of supersymmetric black holes, whose role in this story remains to be understood. One can also show that demanding the absence of naked singularities in (4.4a) implies that $L > \tilde{J}$ [4].

Since the α' corrections are only know in type IIB supergravity, we uplift the solutions (4.4) to ten dimensions. Using the results of [140–142], one can show that Eq. (4.4) oxidises

[3] To avoid cluttering in the notation, from here onward we take $Q \geq 0$ and $J \geq 0$.

[4] These is not the only restrictions on the three-dimensional moduli space of black hole solutions $\{\tilde{J}, \tilde{Q}, r_+\}$ that bulk regularity demands, but it is the only one we will need to show that $\delta S > 0$.

to the following solution of type IIB supergravity:

$$\mathrm{d}s^2 = \mathrm{d}s^2_{5D} + L^2\left[\left(\mathrm{d}\Psi + \mathbb{A} - \frac{A}{\sqrt{3}L}\right)^2 + \mathrm{d}\mathbb{CP}^2\right] \tag{4.11a}$$

$$G_{(5)} = \frac{r^3}{2L}\mathrm{d}t\wedge\mathrm{d}r\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3 + \frac{L^3}{2\sqrt{3}}\,\mathbb{J}\wedge\star_5 F\,, \tag{4.11b}$$

$$F_{(5)} = G_{(5)} + \star_{10}G_{(5)} \tag{4.11c}$$

where $\star_5$ is the five-dimensional Hodge dual obtained using the line element (4.4a), $\star_{10}$ is the Hodge dual obtained using the ten-dimensional line element (4.11a), $\mathrm{d}\mathbb{CP}^2$ is the standard Fubini-Study metric on $\mathbb{CP}^2$ and $\mathbb{J} = \mathrm{d}\mathbb{A}$ is its associated Kähler form.

4.3 Evaluating the corrections

The action[5] with the leading order α' correction is [3]:

$$S_{IIB} = \frac{1}{16\pi G_{10}}\int_{\mathscr{M}_{10}}\mathrm{d}^{10}x\sqrt{-g}\left(R - \frac{1}{4\times 5!}F^2_{(5)} + \gamma\mathscr{W}\right) \tag{4.12}$$

where $\mathscr{W}$ is given by

$$\mathscr{W} \equiv \frac{1}{86016}\sum_{i=1}^{20} n_i M_i \tag{4.13}$$

with all twenty monomials given in table 4.1 and[6]

$$\mathscr{T}_{abcdef} = i\nabla_a F_{bcdef} + \frac{1}{16}\left(F_{abcmn}F_{def}{}^{mn} - 3F_{abfmn}F_{dec}{}^{mn}\right)\,. \tag{4.14}$$

Finally, we also have

$$\gamma = \frac{\alpha'^3}{16}\frac{\pi^3}{8}\zeta(3)\,. \tag{4.15}$$

We notice that table 4.1 corrects some typos in the final table of [3].

Our objective is to use these results to compute the leading correction to the entropy of the black hole solution detailed in (4.4). Naively, one might think that we would need to

[5] As usual, we use this term with a certain abuse of notation, because the five-form $F_{(5)}$ is only made self-dual at the level of the equations of motion. After the inclusion of the correction term proportional to γ, the self-duality condition is accordingly changed.

[6] Note that after computing $\mathscr{T}$ with this expression, one still needs to antisymmetrise over the first three indices and the last three indices and then symmetrise for their exchange, before plugging into the monomials.

n_i	M_i
-43008	$C_{abcd}C_{abef}C_{cegh}C_{dgfh}$
86016	$C_{abcd}C_{aecf}C_{bgeh}C_{dgfh}$
129024	$C_{abcd}C_{aefg}C_{bfhi}\mathcal{T}_{cdeghi}$
30240	$C_{abcd}C_{abce}\mathcal{T}_{dfghij}\mathcal{T}_{efhgij}$
7392	$C_{abcd}C_{abef}\mathcal{T}_{cdghij}\mathcal{T}_{efghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{beghij}\mathcal{T}_{dfghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{bghdij}\mathcal{T}_{eghfij}$
-118272	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dfhgij}$
-26880	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dhifgj}$
112896	$C_{abcd}C_{aefg}\mathcal{T}_{bcfhij}\mathcal{T}_{dehgij}$
-96768	$C_{abcd}C_{aefg}\mathcal{T}_{bcheij}\mathcal{T}_{dfhgij}$
1344	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdeijk}\mathcal{T}_{fghijk}$
-12096	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{eghijk}$
-48384	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{egihjk}$
24192	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cefijk}\mathcal{T}_{dghijk}$
2386	$\mathcal{T}_{abcdef}\mathcal{T}_{abcdgh}\mathcal{T}_{egijkl}\mathcal{T}_{fijhkl}$
-3669	$\mathcal{T}_{abcdef}\mathcal{T}_{abcdgh}\mathcal{T}_{eijgkl}\mathcal{T}_{fikhjl}$
-1296	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dejgkl}\mathcal{T}_{fhkijl}$
10368	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dgjekl}\mathcal{T}_{fhkijl}$
2688	$\mathcal{T}_{abcdef}\mathcal{T}_{abdegh}\mathcal{T}_{cgijkl}\mathcal{T}_{fjkhil}$

Table 4.1 Table detailing the α'^3 corrections of any solution in type IIB supergravity with nontrivial metric g and five-form $F_{(5)}$. Following [3], all tensor monomials are written with all indices lower.

solve the equations of motion from the action (4.12) and only then evaluate the correction to the entropy. However, due to the work in [143] (whose results straightforwardly generalise to the case at hand), one in fact only needs to know the 0^{th} order solution, and evaluate that on the corrected action to get the leading corrections to the entropy.

This is a major simplification and is one of the main reasons this work is possible. However, it is still not a trivial task to evaluate all the monomials from table 4.1 without accidentally inserting typos. Therefore, one of the key steps we had to take was validating our calculations. We wrote two pieces of code independently from one another, only comparing them at the end to make sure they agreed. We started by confirming the results of [3] to make sure there were no mistakes when copying the monomials from table 4.1.

Only after we had two matching codes that confirmed the results in [3] did we insert the solution (4.4). And even then, to be completely certain we had no typos or no convention compatibility issues, not only did we include many consistency checks throughout the code, *e.g.* confirming we indeed solved the correct equations of motion, but we used two

different parametrisations. One of them using a $\mathbb{CP}^2$ fibration and another using a more direct method using the coordinates as originally written in [141]. The $\mathbb{CP}^2$ fibration is the more efficient method and therefore is the one included in the supplemental material. However, the direct method is more amenable to generalisation for the case of different angular momenta [141, 144], which we leave for future work.

After the colossal amount of dust settles, all twenty terms in table 4.1 are non-vanishing on our solutions, and yet the final result appears simple, which gives further confidence in our answer. Using the relation between the Gibbs free energy G and the Euclidean action obtained from (4.12), we find that the stringy correction to the Gibbs free energy at fixed chemical potential μ, angular velocity Ω and temperature T reads

$$(\delta G)_{\mu,\Omega,T} = -\frac{12\pi^3\alpha'^3\left(\tilde{M}+\tilde{Q}\right)^2\zeta(3)}{N^2L^{12}r_+^{15}\left(9L^2-\tilde{J}^2\right)}\left(L^2-\tilde{J}^2\right)^3\Delta\left(\Delta+\frac{4}{L}J\right)\leq 0\,. \tag{4.16}$$

It is a simple matter to compute the variation in entropy, $(\delta S)_{Q,J,M}$, at fixed asymptotic charges Q, J and M from $(\delta G)_{\mu,\Omega,T}$. In particular, we can follow the same steps as in [143] to show that

$$(\delta S)_{Q,J,M} = -T^{-1}(\delta G)_{\mu,\Omega,T}\,. \tag{4.17}$$

Equations (4.16) and (4.17) are the main result of this chapter, whose physical significance we discuss next.

4.4 Interpretation of results

The first thing we note is the fact that $(\delta S)_{Q,J,M}=0$ on the supersymmetric black hole solutions found in [127]. One might wonder why that is the case, given that (4.17) has a factor of T in the denominator, and for supersymmetric solutions $T=0$. However, we note that if we take $\tilde{Q}=\tilde{Q}_{\rm BPS}+\delta Q$ and $\tilde{J}=\tilde{J}_{\rm BPS}+\delta J$, with $\delta Q,\delta J\ll 1$, we get $T=\mathscr{O}(\delta Q,\delta J)$, whereas $\Delta=\mathscr{O}(\delta Q^2,\delta Q\delta J,\delta J^2)$. This means $(\delta S)_{Q,J,M}=\mathscr{O}(\delta Q,\delta J)$ in Eq. (4.17), *i.e.* it vanishes in the supersymmetric limit. Another way to see this result is to note that one can read off the change in entropy due to stringy corrections at constant chemical potential μ, temperature T and angular velocity Ω using the standard thermodynamic relation $S=-(\partial G/\partial T)_{\Omega,\mu}$. In this limit, we get that the correction to the entropy is finite at extremality, being zero in the supersymmetric limit. To our knowledge there is no *a priori* reason, based on bulk physics, for why the entropy in the supersymmetic limit is not corrected via stringy effects. This lends support in favour of the index picture advocated in [111–113, 115–126].

Second, the sign of $(\delta S)_{Q,J,M}$ appears consistent with the weak gravity conjecture [145], similarly to the analogous calculations in flat space [146–150] and with AdS asymptotics [151]. In particular, one can show using the generalisation of the Goon-Penco relation to AdS [150, 151] that the leading correction to the extremality bound at fixed energy M, charge Q and angular momentum J necessarily decreases with respect to the uncorrected solution. This relation is in perfect agreement with the weak gravity conjecture [146, 148, 150].

Thirdly, we point out that our final expression (4.16) only assumes equal angular momenta and equal charges. Notably, it is non-vanishing for a generic non-supersymmetric extremal black hole, and is even valid away from extremality. It would be interesting to understand whether the methods used in [152] could be extended to capture the leading α' corrections presented in this chapter. Further, this then offers a prediction for the quantum field theoretic calculation. Even though the counting of the supersymmetric states is not corrected at finite λ, the counting including non-supersymmetric states should be, and its form should be given by (4.16). However, as of yet, there are no techniques capable of computing a partition function at strong coupling without the aid of supersymmetry. Though we should mention that in [153] some progress has been reported in going slightly beyond the supersymmetric limit.

Our results rely heavily on [143], since we solely use the uncorrected solution to determine the thermodynamic properties of the corrected solution. In principle, we could use the equations of motion that follow from (4.12) together with the modified self-duality condition of [3] to determine directly the stringy corrected black holes. Under such circumstances, we could determine all thermodynamic properties from the solutions *per se* instead of using the arguments presented in [143]. Perhaps our current results suggest that the uncorrected supersymmetric solution might be a solution of the corrected equations of motion. This phenomenon has been recently observed in [154] for a number of corrections and black hole solutions. We leave this avenue of research for the future.

Finally, an interesting avenue for future work is to generalise this calculation to the case when all the angular momenta and charges are distinct, using the results from [144]. The complexity of this solution is quite daunting, and computing these corrections would necessarily require more computing power and a more efficient algorithm[7].

[7]For the interested reader, even just checking that the solution [144] indeed solves the equations of motion as claimed takes a few hours with a rather optimised *Mathematica* code.

Chapter 5

The propagator matrix reloaded

Now we return to the line of reasoning suggested by the second chapter. The development of local RG seems ripe to be used for quantum field theory in curved spacetime calculations. Even without considering AdS/CFT.

But there is a catch. The standard way to perform calculations for quantum field theories involves the S-matrix and the assumption that the theory is free at past and future infinity. However, this assumption may not hold for field theories in non-trivial backgrounds such as curved spacetimes or finite temperature. In fact, even in the simple case of finite temperature Minkowski spacetime, there are a lot of misconceptions and confusion in the literature surrounding how to correctly take interactions into account when setting up the initial conditions.

The objective of this chapter is to clear up these misconceptions and provide a clean and simple derivation of a formalism which includes interactions in the initial conditions and assesses whether or not it is legitimate to ignore them. The ultimate conclusion is that we cannot ignore them: quantum field theories at finite temperature are not free in the infinite past.

This chapter is based on my single-author paper [31].

5.1 Introduction

The S-matrix is the usual object of interest when performing calculations in quantum field theory [32, 40, 155, 156]. It has been extremely successful at reproducing experimental results in particle accelerators but it presents a challenge: in order to construct the 'in' and 'out' aymptotic states we need to assume the theory is asymptotically free at future and past infinity. This is perfectly justified for zero temperature Minkowski spacetime: if we consider local interactions and the 'in' and 'out' states are spatially well separated we do expect the

interactions to die off. However, this might not be the case if we are in the presence of a background field, such as curved spacetime, or are studying a thermal state. In this case the thermal bath and/or the background will keep interacting with the particles possibly ruining our physical picture.

In order to get around these issues we need to calculate new observables, ones which allow us to probe these regimes without assuming the theory is free and checking whether or not our assumptions work. This is precisely what is accomplished by the Schwinger-Keldysh formalism (also sometimes called the 'in-in' formalism) [157, 158]. In this formalism we return to the picture most common in undergraduate quantum mechanics: setting up an initial state at time t_0, evolving up to time t, and calculating the expectation value of the relevant operator. As long as we have control over the initial state, and have the technical prowess to perform the time evolution and evaluate the expectation, there is no need to assume the interactions decay at any time.

This formalism has become a standard tool, being the topic of several textbooks and reviews [33, 159–171]. Using this tool, a lot of attention has been devoted towards studying the situation in the far future. In this case, the main phenomenology is that of secular growth, that is, loop corrections which grow linearly in time, seemlingly ruining perturbation theory at late times. These kinds of issues are well known in the finite temperature literature [161–164, 170, 172–179] and in the case of de Sitter spacetime [7–20]. There have been some calculations performed in black hole and Rindler scenarios [20, 177, 180–184] but the status is less clear in these cases. In order to handle these divergences one needs to construct a modified effective field theory which can take into account the open system character of theories at finite temperature and in the presence of event horizons [7, 8, 11, 12, 15–18]. Studying these divergences was the original motivation for this work and will be the subject of an upcoming publication [185].

However, considerable less attention has been devoted to what happens in the far past. In fact it seems like there is a lot of misunderstanding and confusion in the literature regarding how to appropriately set up initial conditions. Many of the common textbooks and reviews just assume the theory is free at past infinity, essentially ignoring the issue [159, 160, 162, 163, 165–168, 170–173, 177]. Some works are more detailed but end up either changing the dynamics explicitly to turn off the interactions [19, 169, 186, 187] or are based in [188, 189] (for example, [161, 164, 174, 175, 190, 191]) whose arguments have a number flaws which will be discussed in the main body of the chapter and in the conclusion.

The objective of this chapter is to clear up these misconceptions and provide a clean and simple derivation of a formalism which includes interactions in the initial conditions and assesses whether or not it is legitimate to ignore them. The ultimate conclusion is that we

cannot ignore them. There are a number of issues with the standard treatments and explicitly computing the 4-point function one can see that it is never turned off. Quantum field theories at finite temperature are not free in the infinite past.

The manuscript is structured as follows:

In section 5.2, we begin with a brief overview of the Schwinger-Keldysh path integral at a level which should be accessible to readers not familiar with this formalism. We shall pay close attention to the non-triviality of the temporal boundary conditions and the appearance of additional field variables, both of which characteristic of this technique.

In section 5.3, we detail the construction of the Feynman rules for finite temperature initial conditions. We are vary careful about our assumptions and detailed in our reasoning, in particular we shall not assume the interactions are turned off at past infinity and shall set initial conditions at a finite time in the past t_0. The natural conclusion of this calculation is the appearance of a 3×3 propagator matrix, including mixing between the real-time and imaginary-time field variables.

In section 5.4, we continue our analysis by computing the symmetric propagator up to one-loop in an on-shell subtraction scheme. This is correlation function which is necessary to determine the energy-momentum tensor and therefore it has clear physical significance. We pay close attention to the role of the cross terms in our calculation and how the most common approaches in the literature would fail or succeed in obtaining the correct answer.

The conclusion is that the 3×3 approach is more mathematically well-defined and much more straightforward at obtaining the physical answer. However, when resumming the poor IR behaviour of this correlator we find an agreement with the standard approaches. An interpretation for this is provided, nevertheless, this means this calculation is not entirely conclusive on its own regarding the fate of interactions in the far past.

In section 5.5, we settle the question by computing the equal-time 4-point function at tree-level for a particular choice of external momenta. The result is unambiguous: the 3×3 propagator matrix is essential to reproduce the correct answer. Not only is the outcome completely independent of time (which on its own implies the interactions are finite at all times) but also the final answer comes purely from the cross terms.

In section 5.6, we conclude by contrasting with the different approaches found in the literature.

Note: Since submission of the thesis, [192] came out which contradicts some of the claims in section 5.5. More specifically the authors construct a 2×2 formalism which can reproduce (5.57). They achieve this by explicitly turning off the interactions at $t_0 \to -\infty$ by adding a damping factor and then removing this damping at the end. It is worthy of note that in their

formalism one gets different answers depending on whether we set $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = |\boldsymbol{p}_3| = |\boldsymbol{p}_4|$ at the beginning or end of the calculation.

These results suggest the calculations in section 5.5 might not be smoking gun evidence that no 2×2 formalism can reproduce the correct answer. Despite this, no calculations in this chapter are rendered incorrect or invalid with this work. Neither is the interpretation that the interactions do not die off at past infinity, unless of course we force this by hand.

5.2 Review of the Schwinger-Keldysh path integral

In its essence the Schwinger-Keldysh formalism [19, 33, 157–163, 165–173, 186, 190, 193, 194] (also known as 'in-in' formalism) is an initial value formulation of quantum field theory. Instead of considering an 'in' state, $|\text{in}\rangle$, at past infinity and an 'out' state, $|\text{out}\rangle$, at future infinity to then compute the transition amplitude, $S = \langle \text{out}|\text{in}\rangle$; we set up an initial state, $|\psi(t_0)\rangle$, time evolve it, $U(t_f,t_0)\,|\psi(t_0)\rangle$, and then compute the expectation value of some operator $\mathscr{O}(t_f)$:[1]

$$\left\langle \mathscr{O}(t_f)\right\rangle_{\psi} = \langle\psi(t_0)|\,U^{\dagger}(t_f,t_0)\mathscr{O}(t_f)U(t_f,t_0)\,|\psi(t_0)\rangle\,. \tag{5.1}$$

The only difference between this formalism and the usual one is what we are calculating. We can apply this formalism for any theory and any initial state if what we are interested in are expectation values of operators at some time t_f. However, it is worth noting that this formalism is especially useful for time-dependent or out of equilibrium calculations.

In order to perform concrete calculations we need to convert (5.1) to a path integral. To accomplish this we begin by inserting the identity many times[2]:

$$\begin{aligned}&\langle\psi(t_0)|\,U^{\dagger}(t_f,t_0)\mathscr{O}(t_f)U(t_f,t_0)\,|\psi(t_0)\rangle = \\ &= \int \left(\prod \mathrm{d}q_i\right) \langle\psi(t_0)|q_1\rangle\,\langle q_1|\,U^{\dagger}(t_f,t_0)\,|q_2\rangle\,\langle q_2|\,\mathscr{O}(t_f)\,|q_3\rangle\,\langle q_3|\,U(t_f,t_0)\,|q_4\rangle\,\langle q_4|\psi(t_0)\rangle\,.\end{aligned} \tag{5.2}$$

Let us analyse each factor in turn:

[1] We are using the Schrödinger picture, the argument in the operator is an explicit time dependence, not a dynamic/Heisenberg time dependence.

[2] We will sometimes use quantum mechanical notation for simplicity, it should be straightforward to extend to quantum field theories.

- $\langle q_3|\, U(t_f,t_0)\,|q_4\rangle$ is an ordinary path integral with a finite time interval and fixed temporal boundary conditions. The derivation of this fact can be found in standard textbooks and reviews[3] [30, 32, 33, 40, 155, 156, 166]

- $\langle q_1|\, U^\dagger(t_f,t_0)\,|q_2\rangle$ is also an ordinary path integral, however, the presence of the $U^\dagger$ means we are evolving backwards in time from q_2 at t_f to q_1 at t_0. This means we will get an integrand of $\mathrm{e}^{-\mathrm{i}S}$ instead of the more familiar $\mathrm{e}^{\mathrm{i}S}$.

- If our operator of interest is a product of fields (as we shall assume for the remainder of this manuscript) then $\langle q_2|\, \mathscr{O}(t_f)\,|q_3\rangle \propto \delta(q_2-q_3)$, and therefore $q_2=q_3$ and the boundary conditions from our two path integrals match at t_f.

- Finally, $\langle\psi(t_0)|q_1\rangle$ and $\langle q_4|\psi(t_0)\rangle$ are the initial and final wavefunctions. They cannot be readily converted to a path integral. We need to be careful and integrate over all possible boundary conditions at t_0 weighted by these wavefunctions before proceeding. We need to know the functional form of our initial state to perform these calculations.

Putting it all together we get the following path integral:

$$\left\langle \mathscr{O}(t_f)\right\rangle_\rho = \int \mathrm{d}q_-^0\,\mathrm{d}q_+^0\,\rho(q_+^0,q_-^0)\int \mathscr{D}q_+\mathscr{D}q_-\,\mathscr{O}(t_f)\mathrm{e}^{\mathrm{i}S[q_+]-\mathrm{i}S[q_-]} \tag{5.3}$$

with $q_+(t_0)=q_+^0$, $q_-(t_0)=q_-^0$, $q_+(t_f)=q_-(t_f)$ and where we have generalised to an arbitrary density matrix ρ as the above reasoning carries through with no subtleties.

In essence we are starting at time t_0, evolving up to time t_f, inserting the operator of interesting, then evolving backwards towards t_0, integrating over all possible boundary conditions at t_0 weighted by the initial wavefunction. This is sometimes called the 'closed' time contour, however, we should note that it isn't really closed as the fields aren't matched at t_0.

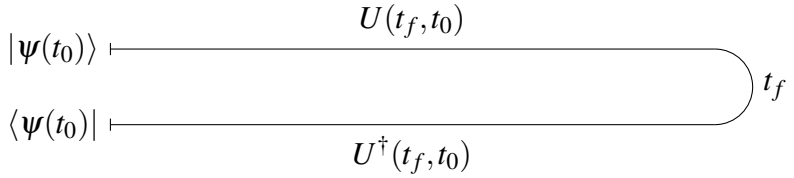


Fig. 5.1 'Closed' time contour

A few remarks are in order. Firstly, that we could insert a $U^\dagger(t_{f_2},t_f)U(t_{f_2},t_f)$ to get either:

[3] And in chapter 2 of this thesis

$$\langle\psi(t_0)|\,U^\dagger(t_{f_2},t_0)U(t_{f_2},t_f)\mathscr{O}(t_f)U(t_f,t_0)\,|\psi(t_0)\rangle \tag{5.4}$$

or

$$\langle\psi(t_0)|\,U^\dagger(t_f,t_0)\mathscr{O}(t_f)U^\dagger(t_{f_2},t_f)U(t_{f_2},t_0)\,|\psi(t_0)\rangle\,. \tag{5.5}$$

Therefore, we can actually insert our operator anywhere on the contour. The time where we turn around and match between the forwards and backwards moving fields is merely a book-keeping parameter and should drop out of the final answer. The physical time variables are t_0 when we set our initial conditions and t_f when we insert the operator.

Secondly, we get a doubling of our field variables. Nevertheless, given the actions are just added together, there seems to be no quadratic mixing and we would naively expect two independent propagators. However, the matching $q_+(t_f) = q_-(t_f)$ actually induces a mixing between the two variables and we get a non-diagonal 2×2 matrix of propagators.

Finally, given we have to integrate over all possible boundary conditions at t_0 we cannot integrate by parts to complete the square as is usual, we have to be a bit more careful. A particularly pedagogical overview of how to perform this for a free theory (including finite temperature and excited states) can be found in [33].

5.3 Tree-level propagators

In this section, we will describe how to construct the Schwinger-Keldysh style path integral, using a finite temperature initial density matrix, set at a finite time in the past, and without assuming the theory to be free at any time. We end by presenting the corresponding Feynman rules for a ϕ^4 theory.

5.3.1 The finite temperature path integral

The finite temperature density matrix is a particularly simple state to construct at any time and without assuming the theory to be free. This is because it is straightforward to convert it to a path integral. We just have to note that the usual Gibbs state (where β is the inverse temperature, H is the Hamiltonian and we have ignored the normalisation as its only role is to cancel the vacuum bubbles):

$$\rho = \mathrm{e}^{-\beta H} \tag{5.6}$$

can be written as a time evolution, albeit in an imaginary direction,

$$\rho = \mathrm{e}^{-\beta H} = U(t_0, t_0 - \mathrm{i}\beta), \tag{5.7}$$

where, for a time dependent Hamiltonian, we should evaluate it at time t_0. This can be readily converted to a Euclidean path integral.

The integration over q_1 and q_2 in (5.2) then implies that the field values are matched along a contour that includes a segment in an imaginary direction as is shown in Fig. 5.2.

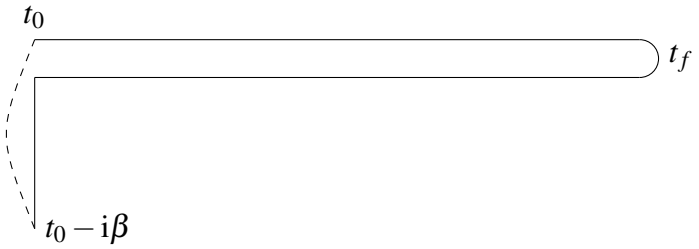


Fig. 5.2 Finite temperature time contour

Our path integral then looks like (for the quantum mechanical theory):

$$Z = \int \mathscr{D}q_+ \mathscr{D}q_- \mathscr{D}q_E \; \mathrm{e}^{\mathrm{i}S[q_+] - \mathrm{i}S[q_-] - S_E[q_E]}, \tag{5.8}$$

where

$$S[q_\pm] = \int_{t_0}^{t_f} \mathrm{d}t \left(\frac{1}{2}\dot{q}_\pm^2 - \frac{1}{2}m^2 q_\pm^2 + J_\pm q_\pm \right), \tag{5.9}$$

$$S_E[q_E] = \int_0^\beta \mathrm{d}\tau \left(\frac{1}{2}{q'_E}^2 + \frac{1}{2}m^2 q_E^2 + J_E q_E \right), \tag{5.10}$$

and where $\tau = -\mathrm{i}t$ is a real parameter for the imaginary segment, $\dot{}$ represents derivatives with respect to t and $'$ derivatives with respect to τ. We have also included sources in anticipation of the calculations to follow and to be more explicit about the sign convention for the factors in front of the sources.

As is clear from the canonical construction for the Schwinger-Keldysh path integral we should impose the following boundary conditions:

$$q_+(t_f) = q_-(t_f), \tag{5.11a}$$

$$q_-(t_0) = q_E(0), \tag{5.11b}$$

$$q_+(t_0) = q_E(\beta). \tag{5.11c}$$

Slightly less obviously we should also impose boundary conditions on the time derivatives of the fields. This will be necessary to solve the propagator equations as they will involve second time derivatives. As is argued in Appendix A of [31] we are free to choose these to be whatever we want. For simplicity we then choose the time derivatives such that all the boundary terms cancel when we integrate by parts:

$$\dot{q}_+(t_f) = \dot{q}_-(t_f), \tag{5.12a}$$
$$\dot{q}_-(t_0) = \mathrm{i} q'_E(0), \tag{5.12b}$$
$$\dot{q}_+(t_0) = \mathrm{i} q'_E(\beta). \tag{5.12c}$$

Note that $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}t}$ which gives some intuition for the factor of i in these equations.

5.3.2 The propagator equations

In order to derive the Feynman rules for ϕ^4 theory in D-dimensional Minkowski spacetime we need to first compute the quadratic path integral including sources. By Fourier transforming in the spatial directions we get the same as in (5.9) where the coefficient in front of the quadratic term is replaced by $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ where $\boldsymbol{p}$ is the spatial momentum, and m is the mass of the particle. Due to this we will continue to use quantum mechanical notation, knowing that it is equivalent to D-dimensional Minkowski spacetime.

Our path integral then looks like, after integrating by parts,

$$\begin{aligned} Z = \int \mathscr{D}q_+ \mathscr{D}q_- \mathscr{D}q_E \exp\Bigg\{ & \mathrm{i}\int_{t_0}^{t_f} \mathrm{d}t \left[-\frac{1}{2} q_+(t) \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + m^2 \right) q_+(t) + J_+(t) q_+(t) \right] - \\ & - \mathrm{i}\int_{t_0}^{t_f} \mathrm{d}t \left[-\frac{1}{2} q_-(t) \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + m^2 \right) q_-(t) + J_-(t) q_-(t) \right] - \\ & - \int_0^{\beta} \mathrm{d}\tau \left[-\frac{1}{2} q_E(\tau) \left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - m^2 \right) q_E(\tau) + J_E(\tau) q_E(\tau) \right] + \\ & + \mathrm{i}\left[\frac{1}{2} q_+(t)\dot{q}_+(t) \right]_{t_0}^{t_f} - \mathrm{i}\left[\frac{1}{2} q_-(t)\dot{q}_-(t) \right]_{t_0}^{t_f} - \left[\frac{1}{2} q_E(\tau) q'_E(\tau) \right]_0^{\beta} \Bigg\}. \end{aligned} \tag{5.13}$$

Now we need to complete the square. We do the following change of variables:

$$Q_+(t_1) = q_+(t_1) + \int_{t_0}^{t_f} \mathrm{d}t_2 \, G_{++}(t_1, t_2) J_+(t_2) - \int_{t_0}^{t_f} \mathrm{d}t_2^\star \, G_{+-}(t_1, t_2^\star) J_-(t_2^\star) +$$

$$+\mathrm{i}\int_{t_0}^{t_f}\mathrm{d}\tau_2\, G_{+E}(t_1,\tau_2)J_E(\tau_2) \tag{5.14a}$$

$$Q_-(t_1^\star) = q_-(t_1^\star) + \int_{t_0}^{t_f}\mathrm{d}t_2\, G_{-+}(t_1^\star,t_2)J_+(t_2) - \int_{t_0}^{t_f}\mathrm{d}t_2^\star\, G_{--}(t_1^\star,t_2^\star)J_-(t_2^\star) + \\ +\mathrm{i}\int_{t_0}^{t_f}\mathrm{d}\tau_2\, G_{-E}(t_1^\star,\tau_2)J_E(\tau_2) \tag{5.14b}$$

$$Q_E(\tau_1) = q_E(\tau_1) + \int_{t_0}^{t_f}\mathrm{d}t_2\, G_{E+}(\tau_1,t_2)J_+(t_2) - \int_{t_0}^{t_f}\mathrm{d}t_2^\star\, G_{E-}(\tau_1,t_2^\star)J_-(t_2^\star) + \\ +\mathrm{i}\int_{t_0}^{t_f}\mathrm{d}\tau_2\, G_{EE}(\tau_1,\tau_2)J_E(\tau_2) \tag{5.14c}$$

note that we include off diagonal terms. This is because the boundary conditions mix the different kind of fields therefore we expect some mixing in the propagator as well. The factors in front of the integrals are mostly conventional but they help match the factors in the integrals for the source terms. The * on the ts are just a convenience to remind which arguments belong to the forwards and backwards time segments.

The propagators need to obey the following equations:

$$\left(-\frac{\partial^2}{\partial t_1^2} - m^2\right) G_{++}(t_1,t_2) = \delta(t_1-t_2) \tag{5.15a}$$

$$\left(-\frac{\partial^2}{\partial t_1^{\star 2}} - m^2\right) G_{-+}(t_1^\star,t_2) = 0 \tag{5.15b}$$

$$\left(-\frac{\partial^2}{\partial \tau_1^2} + m^2\right) G_{E+}(\tau_1,t_2) = 0 \tag{5.15c}$$

$$\left(-\frac{\partial^2}{\partial t_1^2} - m^2\right) G_{+-}(t_1,t_2^\star) = 0 \tag{5.15d}$$

$$\left(-\frac{\partial^2}{\partial t_1^{\star 2}} - m^2\right) G_{--}(t_1^\star,t_2^\star) = -\delta(t_1^\star - t_2^\star) \tag{5.15e}$$

$$\left(-\frac{\partial^2}{\partial \tau_1^2} + m^2\right) G_{E-}(\tau_1,t_2^\star) = 0 \tag{5.15f}$$

$$\left(-\frac{\partial^2}{\partial t_1^2} - m^2\right) G_{+E}(t_1,\tau_2) = 0 \tag{5.15g}$$

$$\left(-\frac{\partial^2}{\partial t_1^{\star 2}} - m^2\right) G_{-E}(t_1^\star,\tau_2) = 0 \tag{5.15h}$$

$$\left(-\frac{\partial^2}{\partial \tau_1^2} + m^2\right) G_{EE}(\tau_1,\tau_2) = -\mathrm{i}\delta(\tau_1-\tau_2) \tag{5.15i}$$

with boundary conditions coming from the field boundary conditions:

$$G_{++}(t_f,t_2) = G_{-+}(t_f,t_2), \quad \left.\frac{\partial G_{++}(t_1,t_2)}{\partial t_1}\right|_{t_1=t_f} = \left.\frac{\partial G_{-+}(t_1^\star,t_2)}{\partial t_1^\star}\right|_{t_1^\star=t_f} \tag{5.16a}$$

$$G_{-+}(t_0,t_2) = G_{E+}(0,t_2), \quad \left.\frac{\partial G_{-+}(t_1^\star,t_2)}{\partial t_1^\star}\right|_{t_1^\star=t_0} = \mathrm{i}\left.\frac{\partial G_{E+}(\tau_1,t_2)}{\partial \tau_1}\right|_{\tau_1=0} \tag{5.16b}$$

$$G_{E+}(\beta,t_2) = G_{++}(t_0,t_2), \quad \mathrm{i}\left.\frac{\partial G_{E+}(\tau_1,t_2)}{\partial \tau_1}\right|_{\tau_1=\beta} = \left.\frac{\partial G_{++}(t_1,t_2)}{\partial t_1}\right|_{t_1=t_0} \tag{5.16c}$$

$$G_{+-}(t_f,t_2^\star) = G_{--}(t_f,t_2^\star), \quad \left.\frac{\partial G_{+-}(t_1,t_2^\star)}{\partial t_1}\right|_{t_1=t_f} = \left.\frac{\partial G_{--}(t_1^\star,t_2^\star)}{\partial t_1^\star}\right|_{t_1^\star=t_f} \tag{5.16d}$$

$$G_{--}(t_0,t_2^\star) = G_{E-}(0,t_2^\star), \quad \left.\frac{\partial G_{--}(t_1^\star,t_2^\star)}{\partial t_1^\star}\right|_{t_1^\star=t_0} = \mathrm{i}\left.\frac{\partial G_{E-}(\tau_1,t_2^\star)}{\partial \tau_1}\right|_{\tau_1=0} \tag{5.16e}$$

$$G_{E-}(\beta,t_2^\star) = G_{+-}(t_0,t_2^\star), \quad \mathrm{i}\left.\frac{\partial G_{E-}(\tau_1,t_2)}{\partial \tau_1}\right|_{\tau_1=\beta} = \left.\frac{\partial G_{+-}(t_1,t_2^\star)}{\partial t_1}\right|_{t_1=t_0} \tag{5.16f}$$

$$G_{+E}(t_f,\tau_2) = G_{-E}(t_f,\tau_2), \quad \left.\frac{\partial G_{+E}(t_1,\tau_2)}{\partial t_1}\right|_{t_1=t_f} = \left.\frac{\partial G_{+E}(t_1,\tau_2)}{\partial t_1}\right|_{t_1=t_f} \tag{5.16g}$$

$$G_{-E}(t_0,\tau_2) = G_{EE}(0,t_2), \quad \left.\frac{\partial G_{-E}(t_1^\star,\tau_2)}{\partial t_1^\star}\right|_{t_1^\star=t_0} = \mathrm{i}\left.\frac{\partial G_{EE}(\tau_1,\tau_2)}{\partial \tau_1}\right|_{\tau_1=0} \tag{5.16h}$$

$$G_{EE}(\beta,\tau_2) = G_{+E}(t_0,\tau_2), \quad \mathrm{i}\left.\frac{\partial G_{EE}(\tau_1,\tau_2)}{\partial \tau_1}\right|_{\tau_1=\beta} = \left.\frac{\partial G_{+E}(t_1,\tau_2)}{\partial t_1}\right|_{t_1=t_0} \tag{5.16i}$$

so that $Q_\pm$ and Q_E have vanishing boundary conditions. They are ordered them in this particular way to highlight that even though they are nine coupled equations they come in three cycles of three equations each. Also note that the boundary conditions are only imposed in the first argument, the only way the two arguments mix is via the delta functions in the diagonal components. There is a diagonal component in each set so all equations end up mixing the two arguments.

After these simplifications it is fairly straightforward to solve the equations to get:

$$\begin{aligned} G_{++}(t_1,t_2) = &-\frac{\mathrm{i}}{2m}\cos\left(m\left(t_1-t_2-\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right)- \\ &-\frac{1}{m}\Theta(t_1-t_2)\sin(m(t_1-t_2)) \end{aligned} \tag{5.17a}$$

$$\begin{aligned} G_{--}(t_1^\star,t_2^\star) = &-\frac{\mathrm{i}}{2m}\cos\left(m\left(t_1^\star-t_2^\star+\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right)+ \\ &+\frac{1}{m}\Theta(t_1^\star-t_2^\star)\sin(m(t_1^\star-t_2^\star)) \end{aligned} \tag{5.17b}$$

$$G_{EE}(\tau_1,\tau_2) = -\frac{\mathrm{i}}{2m}\cosh\left(m\left(\tau_1-\tau_2+\frac{\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right)+ \\ +\frac{1}{m}\Theta(\tau_1-\tau_2)\sinh(m(\tau_1-\tau_2)) \tag{5.17c}$$

$$G_{+-}(t_1,t_2^\star) = -\frac{\mathrm{i}}{2m}\cos\left(m\left(t_1-t_2^\star-\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) = G_{-+}(t_2^\star,t_1) \tag{5.17d}$$

$$G_{+E}(t_1,\tau_2) = -\frac{\mathrm{i}}{2m}\cos\left(m\left(t_1-t_0+\mathrm{i}\tau_2-\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) = G_{E+}(\tau_2,t_1) \tag{5.17e}$$

$$G_{-E}(t_1^\star,\tau_2) = -\frac{\mathrm{i}}{2m}\cos\left(m\left(t_1^\star-t_0+\mathrm{i}\tau_2-\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) = G_{E-}(\tau_2,t_1^\star) \tag{5.17f}$$

Symmetrising the diagonal components and inserting $1 = \Theta(t_1-t_2)+\Theta(t_2-t_1)$ we get:

$$G_{++}^{\mathrm{sym}}(t_1,t_2) = -\frac{\mathrm{i}}{2m}\cos\left(m\left(|t_1-t_2|+\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) \tag{5.18a}$$

$$G_{--}^{\mathrm{sym}}(t_1^\star,t_2^\star) = -\frac{\mathrm{i}}{2m}\cos\left(m\left(|t_1^\star-t_2^\star|-\frac{\mathrm{i}\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) \tag{5.18b}$$

$$G_{EE}^{\mathrm{sym}}(\tau_1,\tau_2) = -\frac{\mathrm{i}}{2m}\cosh\left(m\left(|\tau_1-\tau_2|-\frac{\beta}{2}\right)\right)\operatorname{csch}\left(\frac{m\beta}{2}\right) \tag{5.18c}$$

We now have nine propagators which seem largely independent. Nevertheless, there are some symmetries that can be exploited to reduce the number of propagators we actually have to consider. This is accomplished by changing to the average-difference basis, also called the Keldysh basis [33, 159–165, 168, 170, 190, 193, 195].

We define,

$$J_{\mathrm{ave}} = \frac{J_+ + J_-}{2}, \quad J_{\mathrm{dif}} = J_+ - J_- \tag{5.19a}$$

$$q_{\mathrm{ave}} = \frac{q_+ + q_-}{2}, \quad q_{\mathrm{dif}} = q_+ - q_- \tag{5.19b}$$

Plugging this into the above and using the fact that

$$G_{++}^{\mathrm{sym}}(t_1,t_2) + G_{--}^{\mathrm{sym}}(t_1,t_2) = G_{+-}(t_1,t_2) + G_{-+}(t_1,t_2) \tag{5.20}$$

we get

$$Z = \exp\Bigg\{ -\frac{\mathrm{i}}{2}\int \mathrm{d}t_1\,\mathrm{d}t_2\, J_{\mathrm{dif}}(t_1)G_{\mathrm{ave,ave}}(t_1,t_2)J_{\mathrm{dif}}(t_2) - \\ -\frac{\mathrm{i}}{2}\int \mathrm{d}t_1\,\mathrm{d}t_2\, J_{\mathrm{ave}}(t_1)G_{\mathrm{dif,ave}}(t_1,t_2)J_{\mathrm{dif}}(t_2) -$$

$$
\begin{aligned}
&-\frac{\mathrm{i}}{2}\int \mathrm{d}t_1\,\mathrm{d}t_2\, J_{\mathrm{dif}}(t_1)G_{\mathrm{ave,dif}}(t_1,t_2)J_{\mathrm{ave}}(t_2)+\\
&+\frac{1}{2}\int \mathrm{d}t_1\,\mathrm{d}\tau_2\, J_{\mathrm{dif}}(t_1)G_{\mathrm{ave,E}}(t_1,\tau_2)J_{\mathrm{E}}(\tau_2)+\\
&+\frac{1}{2}\int \mathrm{d}\tau_1\,\mathrm{d}t_2\, J_{\mathrm{E}}(\tau_1)G_{\mathrm{E,ave}}(\tau_1,t_2)J_{\mathrm{dif}}(t_2)+\\
&+\frac{\mathrm{i}}{2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\, J_{\mathrm{E}}(\tau_1)G_{\mathrm{E,E}}(\tau_1,\tau_2)J_{\mathrm{E}}(\tau_2)\Bigg\}
\end{aligned}
\tag{5.21}
$$

where

$$G_{\mathrm{ave,ave}}(t_1,t_2) = -\frac{\mathrm{i}}{2m}\cos(m(t_1-t_2))\coth\left(\frac{m\beta}{2}\right) \tag{5.22a}$$

$$G_{\mathrm{dif,ave}}(t_1,t_2) = \frac{1}{m}\sin(m(t_1-t_2))\Theta(t_2-t_1) = G_{\mathrm{ave,dif}}(t_2,t_1) \tag{5.22b}$$

$$G_{\mathrm{ave,E}}(t_1,\tau_2) = G_{+,E}(t_1,\tau_2) = G_{-E}(t_1,\tau_2) = G_{\mathrm{E,ave}}(\tau_2,t_1) \tag{5.22c}$$

Note that the $J_{\mathrm{ave}}J_{\mathrm{ave}}$ and the $J_{\mathrm{ave}}J_E$ terms vanish identically. Also note that we have labelled the propagators so that any 'dif' label is together with a J_{ave} and vice-versa, this is on purpose because

$$J_+q_+ - J_-q_- = J_{\mathrm{ave}}q_{\mathrm{dif}} + J_{\mathrm{dif}}q_{\mathrm{ave}} \tag{5.23}$$

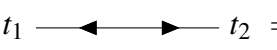
With this convention the 'dif' and 'ave' labels on diagrams will coincide with that will appear in correlators as functions of fields and with what appears in the potential.

5.3.3 Feynman rules in the average-difference basis

To deduce the Feynman rules we have to be careful with factors of i and -1 due to the mixing between real and imaginary fields, in Appendix B of [31] we present the derivation, in the main text we will just present the result.

For the average-difference basis in particular, since in $G_{\mathrm{dif,ave}}(t_1,t_2)$ we know that $t_2 > t_1$ we will draw an arrow from 'dif' to 'ave'. This flow implied by the arrows is usually called 'causal flow' because it tells you the direction of time. It is straightforward to see we cannot have a closed 'causal' loop, because we would have products of Heaviside-Θs that would always vanish. The other propagators do not have any causal connections but for ease of

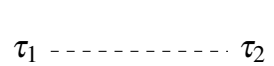
visibility there will always be arrows pointing towards a 'ave' end and legs that connect with Euclidean times will be dashed. In summary, here's the notation we shall use[4]:



$$= \mathrm{i}G_{\mathrm{ave,ave}}(t_1,t_2) \tag{5.24a}$$

$$= \mathrm{i}G_{\mathrm{dif,ave}}(t_1,t_2) \tag{5.24b}$$

$$= \mathrm{i}G_{\mathrm{ave,E}}(t_1,\tau_2) \tag{5.24c}$$

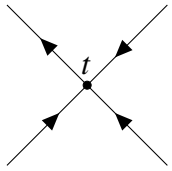


$$= \mathrm{i}G_{\mathrm{E,E}}(\tau_1,\tau_2) \tag{5.24d}$$

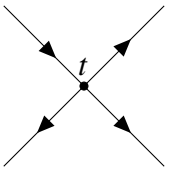
In terms of vertices, there are three kinds. We have a quartic Euclidean vertex, and two Lorentzian ones. Since

$$\frac{1}{4!}q^4 - \frac{1}{4!}q'^4 = \frac{q_{\mathrm{ave}}^3}{3!}q_{\mathrm{dif}} + \frac{1}{4}q_{\mathrm{ave}}\frac{q_{\mathrm{dif}}^3}{3!} \tag{5.25}$$

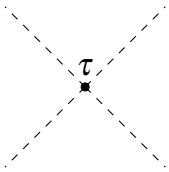
there is one Lorentzian vertex with three 'ave' and one 'dif' and another with three 'dif' and one 'ave'. Because there are only three identical legs in these vertices, the vertex with three 'dif' comes with an additional factor of $\frac{1}{4}$. In summary, we have:



$$= -\mathrm{i}\lambda \int \mathrm{d}t \tag{5.26a}$$



$$= -\mathrm{i}\frac{\lambda}{4} \int \mathrm{d}t \tag{5.26b}$$



$$= -\lambda \int \mathrm{d}\tau \tag{5.26c}$$

where in the last rule the dashed external legs may also have arrows if they come form a $G_{\mathrm{ave,E}}$.

[4]Note how, due to the proliferation of different types of propagators this chapter uses a different convention regarding dashed and continuous lines than the previous chapters.

In higher dimensions all of the propagators also carry a momentum label. We should proceed exactly as in ordinary Feynman rules, we impose momentum conservation along propagators and vertices, and we integrate over loop momenta. Throughout the chapter we shall drop overall momentum conserving Dirac-δs for ease of notation.

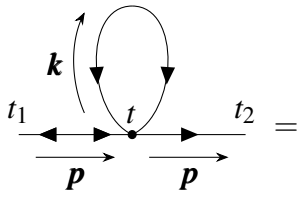
5.4 One-Loop symmetric propagator

We now compute the symmetric 2-point function $\langle\{\phi(x_1),\phi(x_2)\}\rangle$. In the average-difference basis it becomes:

$$\begin{aligned}\langle\{\phi(x_1),\phi(x_2)\}\rangle &= \langle\phi_+(x_1)\phi_-(x_2)+\phi_-(x_1)\phi_+(x_2)\rangle = \\ &= \left\langle\left(\phi_{\rm ave}(x_1)+\frac{\phi_{\rm dif}(x_2)}{2}\right)\left(\phi_{\rm ave}(x_2)-\frac{\phi_{\rm dif}(x_1}{2}\right)+\right. \\ &\quad \left.+\left(\phi_{\rm ave}(x_1)-\frac{\phi_{\rm dif}(x_2)}{2}\right)\left(\phi_{\rm ave}(x_2)+\frac{\phi_{\rm dif}(x_1)}{2}\right)\right\rangle \\ &= \left\langle 2\phi_{\rm ave}(x_1)\phi_{\rm ave}(x_2)-\frac{1}{2}\phi_{\rm dif}(x_1)\phi_{\rm dif}(x_2)\right\rangle \end{aligned} \tag{5.27}$$

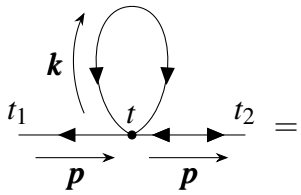
where in the first line we have forced the ordering by placing one of the field operators in the forward moving segment (which appears first in the time contour) and the other on the backwards moving segment (which appears later in the contour). Also note that the last term in (5.27) vanishes (at least up to one-loop).

The diagrams that contribute to the symmetric 2-point function at 1-loop level are:

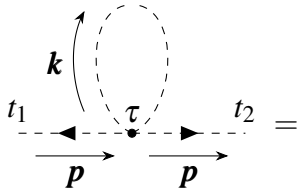


$$\begin{aligned}&= -\frac{\mathrm{i}\lambda}{2}\int_{t_0}^{t_f}\mathrm{d}t\int\frac{\mathrm{d}^dk}{(2\pi)^d}\mathrm{i}G_{\rm ave,ave}(\boldsymbol{p},t_1,t)\mathrm{i}G_{\rm ave,ave}(\boldsymbol{k},t,t)\mathrm{i}G_{\rm dif,ave}(\boldsymbol{p},t,t_2) = \\ &= -\frac{\lambda}{2}\int_{t_0}^{t_f}\mathrm{d}t\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{2E_{\boldsymbol{p}}}\cos[E_{\boldsymbol{p}}(t_1-t)]\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\frac{-\mathrm{i}}{2E_{\boldsymbol{k}}}\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)\cdot \\ &\quad\cdot\frac{1}{E_{\boldsymbol{p}}}\sin[E_{\boldsymbol{p}}(t-t_2)]\Theta(t_2-t) = \end{aligned}$$

$$=\frac{\lambda}{32E_{\boldsymbol{p}}^3}\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(\cos[E_{\boldsymbol{p}}(t_1+t_2-2t_0)]-\cos[E_{\boldsymbol{p}}(t_1-t_2)]+2E_{\boldsymbol{p}}(t_2-t_0)\sin[E_{\boldsymbol{p}}(t_1-t_2)]\right)\cdot$$
$$\cdot\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)}{E_{\boldsymbol{k}}} \tag{5.28}$$



$$=-\frac{\mathrm{i}\lambda}{2}\int_{t_0}^{t_f}\mathrm{d}t\int\frac{\mathrm{d}^dk}{(2\pi)^d}\mathrm{i}G_{\text{ave,dif}}(\boldsymbol{p},t_1,t)\mathrm{i}G_{\text{ave,ave}}(\boldsymbol{k},t,t)\mathrm{i}G_{\text{ave,ave}}(\boldsymbol{p},t,t_2)=$$
$$=-\frac{\lambda}{2}\int_{t_0}^{t_f}\mathrm{d}t\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{E_{\boldsymbol{p}}}\sin[E_{\boldsymbol{p}}(t-t_1)]\Theta(t_1-t)\frac{-\mathrm{i}}{2E_{\boldsymbol{k}}}\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)\cdot$$
$$\cdot\frac{-\mathrm{i}}{2E_{\boldsymbol{p}}}\cos[E_{\boldsymbol{p}}(t-t_2)]\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)=$$
$$=\frac{\lambda}{32E_{\boldsymbol{p}}^3}\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(\cos[E_{\boldsymbol{p}}(t_1+t_2-2t_0)]-\cos[E_{\boldsymbol{p}}(t_1-t_2)]-2E_{\boldsymbol{p}}(t_1-t_0)\sin[E_{\boldsymbol{p}}(t_1-t_2)]\right)\cdot$$
$$\cdot\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)}{E_{\boldsymbol{k}}} \tag{5.29}$$



$$=-\frac{\lambda}{2}\int_0^{\beta}\mathrm{d}\tau\int\frac{\mathrm{d}^dk}{(2\pi)^d}\mathrm{i}G_{\text{ave,E}}(\boldsymbol{p},t_1,\tau)\mathrm{i}G_{E,E}(\boldsymbol{k},\tau,\tau)\mathrm{i}G_{\text{E,ave}}(\boldsymbol{p},\tau,t_2)=$$
$$=\frac{\mathrm{i}\lambda}{2}\int_0^{\beta}\mathrm{d}\tau\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{2E_{\boldsymbol{p}}}\cos\left[E_{\boldsymbol{p}}\left(t_1-t_0+\mathrm{i}\tau-\mathrm{i}\frac{\beta}{2}\right)\right]\operatorname{csch}\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\frac{-\mathrm{i}}{2E_{\boldsymbol{k}}}\cosh\left(-\frac{E_{\boldsymbol{k}}\beta}{2}\right)\operatorname{csch}\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)\cdot$$
$$\cdot\frac{-\mathrm{i}}{2E_{\boldsymbol{p}}}\cos\left[E_{\boldsymbol{p}}\left(t_2-t_0+\mathrm{i}\tau-\mathrm{i}\frac{\beta}{2}\right)\right]\operatorname{csch}\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)=$$
$$=-\frac{\lambda}{32E_{\boldsymbol{p}}^3}\left(\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\cos\left[E_{\boldsymbol{p}}(t_1+t_2-2t_0)\right]\right)\cdot$$

$$\cdot \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)}{E_{\boldsymbol{k}}} \tag{5.30}$$

notice how in this diagram we have contributions which do not depend on t_0. Therefore, even for initial conditions set in the infinite past you need to include these cross terms.

Adding it all up we get:

$$\begin{aligned}
&\left\langle\left\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\right\}\right\rangle_{\text{1-loop}} = 2\left\langle\tilde{\phi}_{\text{ave}}(\boldsymbol{p},t_1)\tilde{\phi}_{\text{ave}}(-\boldsymbol{p},t_2)\right\rangle_{\text{1-loop}} = \\
&= -\frac{\lambda}{16E_{\boldsymbol{p}}^3}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)}{E_{\boldsymbol{k}}}\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\right.\right. \\
&\left.\left.+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)
\end{aligned} \tag{5.31}$$

Note that the t_0 dependence cancelled between the three diagrams as is to be expected from the time-translation invariance of the thermal state.

We still need to add the counterterms. Usually we need to resum the series to consider 1PI graphs [30, 32, 40, 155], but this is much harder in this formalism, so what we shall do instead is to make $m^2 \to m^2 + \delta m^2$ in the tree-level answer and expand in powers of δm^2. The idea is that δm^2 is linear in λ. This is actually a bit closer to the spirit of renormalisation, we are figuring out what is the function $m^2(\lambda,\Lambda)$ that we need to put in the action so that m^2 corresponds to the physical measured mass (squared) and then expanding in powers of λ (Λ is the cutoff, we'll be mostly agnostic about how exactly we are regulating the theory). We then get:

$$\begin{aligned}
&-\frac{\mathrm{i}\coth\left(\frac{1}{2}\beta\sqrt{m^2+\delta m^2+\boldsymbol{p}^2}\right)\cos\left[(t_1-t_2)\sqrt{m^2+\delta m^2+\boldsymbol{p}^2}\right]}{2\sqrt{m^2+\delta m^2+\boldsymbol{p}^2}} = \\
&= -\frac{\mathrm{i}}{2E_{\boldsymbol{p}}}\cos(E_{\boldsymbol{p}}(t_1-t_2))\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)+ \\
&+\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right])+\right. \\
&\left.+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)\frac{\mathrm{i}\delta m^2}{8E_{\boldsymbol{p}}^3}+O(\delta m^2)^2
\end{aligned} \tag{5.32}$$

The contribution to the symmetric 2-point function at $O(\lambda)$ is then:

$$\left\langle\left\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\right\}\right\rangle_{\delta m^2} = -\frac{\delta m^2}{4E_{\boldsymbol{p}}^3}\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\right.\right.$$
$$\left.\left.+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right) \tag{5.33}$$

Similarly, there is also the question of field renormalisation. In the same vein as above, what we need to do is insert a $Z(\lambda,\Lambda)$ as a coefficient to the kinetic term, expand in powers of λ and figure out what is the physical normalisation. This avoids dealing with diagrams with time derivatives. Naively it seems like we need to solve the equations once again, however, by looking at the derivation of (5.15) we see that adding Z would correspond to multiplying the $\frac{\partial^2}{\partial t^2}$ terms by Z. However, if we define $m'^2=\frac{m^2}{Z}$ and $G'=ZG$ then G' solves the same equation as if we had no field renormalisation since the boundary conditions don't depend on the normalisation of G. Therefore, we have:

$$G'_{\text{ave,ave}}(\boldsymbol{p},t_1,t_2) = -\frac{\mathrm{i}\coth\left(\frac{1}{2}\beta\sqrt{m'^2+\boldsymbol{p}^2}\right)\cos\left[(t_1-t_2)\sqrt{m'^2+\boldsymbol{p}^2}\right]}{2\sqrt{m'^2+\boldsymbol{p}^2}} \Leftrightarrow$$
$$\Leftrightarrow G_{\text{ave,ave}}(\boldsymbol{p},t_1,t_2) = -\frac{\mathrm{i}\coth\left(\frac{1}{2}\beta\sqrt{\frac{m^2}{Z}+\boldsymbol{p}^2}\right)\cos\left[(t_1-t_2)\sqrt{\frac{m^2}{Z}+\boldsymbol{p}^2}\right]}{2Z\sqrt{\frac{m^2}{Z}+\boldsymbol{p}^2}} \tag{5.34}$$

Now expanding in powers of λ as $Z=1+\delta Z$ we get:

$$= -\frac{\mathrm{i}}{2E_{\boldsymbol{p}}}\cos(E_{\boldsymbol{p}}(t_1-t_2))\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)-$$
$$-\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\right.\right.$$
$$+\left.\left(1-\frac{2E_{\boldsymbol{p}}^2}{m^2}\right)\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)+$$
$$\left.+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)\frac{\mathrm{i}m^2\delta Z}{8E_{\boldsymbol{p}}^3}+O(\delta Z)^2 \tag{5.35}$$

which is very similar to the mass counterterm, except it has an additional term.

The full 1-loop contribution to the symmetric 2-point function including counterterms is:

$$\begin{aligned}
&\left\langle\left\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\right\}\right\rangle_{\text{1-loop+c.t.}} = \\
&= -\frac{\lambda I_\beta(\Lambda)+4\delta m^2-4m^2\delta Z}{16E_{\boldsymbol{p}}^3}\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\right.\right. \\
&\left.\left.+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)-\frac{\delta Z}{E_{\boldsymbol{p}}}\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]
\end{aligned} \tag{5.36}$$

where

$$I_\beta(\Lambda)=\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)}{E_{\boldsymbol{k}}} \tag{5.37}$$

and the integral is assumed to be regulated in some way.

5.4.1 Choice of counteterms

In order to choose an appropriate δm^2 and δZ we need some physical definition of mass and field renormalisation. Given these are parameters in the action/Hamiltonian we do not expect them to depend on the temperature. For example, if the mass is defined as the energy gap in the spectrum, this will be a feature of the Hamiltonian rather than of the initial state we put our system in. This means we should take the zero temperature limit and then use the usual Källén-Lehmann spectral representation [40, 155] to get an appropriate definition of mass and field renormalisation.

The $\beta\to\infty$ limit of the above reads

$$\begin{aligned}
&\left\langle\left\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\right\}\right\rangle^{\beta\to\infty}_{\text{1-loop+c.t.}} = \langle\Omega|\left\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\right\}|\Omega\rangle_{\text{1-loop+c.t.}} = \\
&= -\frac{\lambda I_\infty(\Lambda)+4\delta m^2-4m^2\delta Z}{8E_{\boldsymbol{p}}^3}\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)- \\
&\quad -\frac{\delta Z}{E_{\boldsymbol{p}}}\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]
\end{aligned} \tag{5.38}$$

where

$$I_\infty(\Lambda)=\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{E_{\boldsymbol{k}}} \tag{5.39}$$

and $|\Omega\rangle$ is defined as the ground state of the Hamiltonian (in principle at time t_0). In the limit $\beta\to\infty$ this is the only state that contributes.

By running the usual arguments for the Källén-Lehmann spectral representation [40, 155] but for the symmetric 2-point function we get

$$\langle\Omega|\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\}|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\frac{\rho(M^2)}{E_{\boldsymbol{p}}}\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right] \tag{5.40}$$

by setting $\boldsymbol{p}=\boldsymbol{0}$, $t_2=0$, and $t_1=t$ to simplify our calculations (ρ cannot depend on any of these variables by construction) it is straightforward to get

$$\begin{aligned}\rho(M^2) =& \left(1-\frac{\lambda I_\infty(\Lambda)+4\delta m^2-4m^2\delta Z}{8m^2}-\delta Z\right)2\pi\delta(M^2-m^2)+\\ &+\left(\frac{\lambda}{4}I_\infty(\Lambda)+\delta m^2-m^2\delta Z\right)2\pi\frac{\partial}{\partial M^2}\left(\delta(M^2-m^2)\right)\end{aligned} \tag{5.41}$$

This seems like a bit of a weird behaviour since we get a delta function at m^2 but we also get a derivative of a delta function, which is more singular than would be expected. However, this is just an artefact of our perturbative expansion. In fact, this expression is equivalent to shifting the pole by an amount

$$\Delta = \frac{\lambda}{4}I_\infty(\Lambda)+2\delta m^2-2m^2\delta Z \tag{5.42}$$

that is, we can also write $\rho(M^2)$ as

$$\rho(M^2) = \left(1-\frac{\Delta}{2m^2}-\frac{\delta Z}{2}\right)2\pi\delta(M^2-m^2+\Delta) \tag{5.43}$$

and obtain the previous answer by expanding in powers of λ, δm^2, and δZ.

Our physical renormalisation conditions (choosing m^2 to be our physical mass) are that the pole is at m^2 and that the coefficient in front is 1. Solving for the counterterms we get:

$$\delta m^2 = -\frac{\lambda}{4}I_\infty(\Lambda) \tag{5.44a}$$

$$\delta Z = 0 \tag{5.44b}$$

The end result is then:

$$\left\langle\{\tilde{\phi}(\boldsymbol{p},t_1),\tilde{\phi}(-\boldsymbol{p},t_2)\}\right\rangle_{\text{1-loop+c.t.}} =$$

$$= -\frac{\lambda\left(I_\beta(\Lambda)-I_\infty(\Lambda)\right)}{16E_{\boldsymbol{p}}^3}\left(2\coth\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)\left(E_{\boldsymbol{p}}(t_1-t_2)\sin\left[E_{\boldsymbol{p}}(t_1-t_2)\right]+\right.\right.$$
$$\left.\left.+\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right)+\operatorname{csch}^2\left(\frac{E_{\boldsymbol{p}}\beta}{2}\right)E_{\boldsymbol{p}}\beta\cos\left[E_{\boldsymbol{p}}(t_1-t_2)\right]\right) \tag{5.45}$$

Note that the integral

$$\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\coth\left(\frac{E_{\boldsymbol{k}}\beta}{2}\right)-1}{E_{\boldsymbol{k}}} \tag{5.46}$$

is convergent even without a cutoff. With a finite cutoff it depends on the cutoff but that dependence is negligible if the cutoff is far above any scales of interest. This behaviour is exactly what is expected of a field theory at finite temperature [157–160, 162, 163, 165, 170, 172, 173, 176, 196, 197].

The final answer does not contain any terms proportional to t_1+t_2 therefore there are no secular effects. However, there is still a temporal IR growth from the term proportional to (t_1-t_2). This does not affect the energy-momentum tensor (as it vanishes in the coincidence limit) but it means that naive perturbation theory is inadequate if the temporal separation is too large. However, this effect is easy to resum.

First note that if instead we chose a temperature dependent counterterm:

$$\delta m^2 = -\frac{\lambda}{4}I_\beta \tag{5.47}$$

the mass parameter would not correspond to the physical mass as it won't be the energy gap in the spectrum, but the secular effect won't be there. It is also not very physical to have terms in the Hamiltonian that depend on the choice of initial conditions[5]. However, this tells us how to resum these terms.

Then note that the physical choice of counterterm means that the relation between the physical mass m^2_{phys} and the mass parameter in the Lagrangian $m^2_{\text{Lagrangian}}$ is

$$m^2_{\text{Lagrangian}} = m^2_{\text{phys}} - \frac{\lambda}{4}I_\infty \tag{5.48}$$

where m_{phys} is independent of the regulator.

[5]The author thanks Stefan Hollands for pointing this out.

All in all this suggests that if we insert as a mass parameter in the propagators:

$$m^2_{\text{prop}} = m^2_{\text{Lagrangian}} + \frac{\lambda}{4} I_\beta = m^2_{\text{phys}} + \frac{\lambda}{4}(I_\beta - I_\infty) \tag{5.49}$$

then we rescue perturbation theory at large temporal separations. Note that we are not inserting this in the Lagrangian, the claim is that the contribution from these diagrams could be resumed by using this modified propagator. This agrees with what is found in the literature for the thermal mass shift [157–160, 162, 163, 165, 170, 172, 173, 176, 196, 197].

Had we taken the naive approach and not considered the $G_{\text{ave,E}}$ cross terms we would have found several issues. Firstly, we would find that the final answer depends on t_0. This is to be expected, by disregarding these terms we are essentially taking $\rho = \exp(-\beta H_0)$ as our initial state, where H_0 is the free part of the Hamiltonian. Given the free Hamiltonian does not commute with the full Hamiltonian we ought to expect time dependence. However, this time dependence is not ameliorated by taking the limit $t_0 \to -\infty$ as the dependence is oscillatory rather than decaying. We could perhaps take the limit in such a way to turn those oscillations into damping [174, 187] however, we would then not recover the final term that arises from the cross terms which puts this method into question.

However, there is some evidence that in some sense $\rho = \exp(-\beta H_0)$ is 'close enough' to the desired state. Had we only included the 2×2 propagators and only included the counterterms in the interaction Hamiltonian rather than expanding the tree-level propagator as we did, we would obtain the correct IR resummation. This suggests there could be some dynamical effect which makes the two states agree once we fix their IR behaviour. Nevertheless, this claim relies on the fact this resummation would continue to agree at every loop level, which, to the knowledge of the author, has not been proven.

Further, we would have obtained a different answer depending on whether we do counterterms as usual (which corresponds to inserting them in the interaction Hamiltonian) or expanding the tree-level propagator (which corresponds to inserting them in the free Hamiltonian). This difference arises because the initial state depends on the free Hamiltonian but not the interaction Hamiltonian. This puts into question the mathematical consistency of the whole formalism.

To fully settle the debate, in the next section we explicitly calculate the equal-time 4-point function, checking whether or not it would be possible to get an agreement between the various approaches. Once more this is a very physical quantity to calculate as it is often the object of interest in, *e.g.* cosmological applications [7–20].

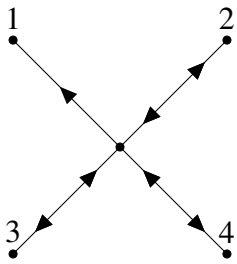
5.5 Tree-level equal-time 4-point function

We wish to calculate:

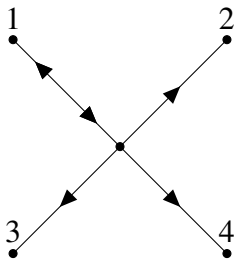
$$\begin{aligned}&\left\langle\tilde{\phi}(\boldsymbol{p}_1,t_f)\tilde{\phi}(\boldsymbol{p}_2,t_f)\tilde{\phi}(\boldsymbol{p}_3,t_f)\tilde{\phi}(\boldsymbol{p}_4,t_f)\right\rangle_\beta = \\ &=\left\langle\tilde{\phi}_{\text{ave}}(\boldsymbol{p}_1,t_f)\tilde{\phi}_{\text{ave}}(\boldsymbol{p}_2,t_f)\tilde{\phi}_{\text{ave}}(\boldsymbol{p}_3,t_f)\tilde{\phi}_{\text{ave}}(\boldsymbol{p}_4,t_f)\right\rangle_\beta\end{aligned} \tag{5.50}$$

where, in the last line, we used the fact that the equal-time means we can use $\phi_\pm$ interchangeably and therefore we can use ϕ_{ave}.

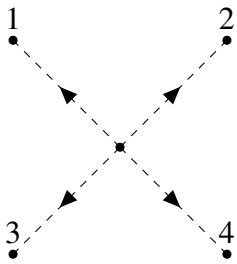
The diagrams that contribute are:



+ permutations (5.51)



+ permutations (5.52)



+ permutations (5.53)

Let's choose $E_{\boldsymbol{p}_1} = E_{\boldsymbol{p}_2} = E_{\boldsymbol{p}_3} = E_{\boldsymbol{p}_4} = E$, or equivalently $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = |\boldsymbol{p}_3| = |\boldsymbol{p}_4|$ for simplicity, then

$$\begin{aligned}(5.51) = &-\mathrm{i}\lambda\int_{t_0}^{t_f}\mathrm{d}t\,\mathrm{i}G_{\text{dif,ave}}(\boldsymbol{p}_1,t,t_f)\mathrm{i}G_{\text{ave,ave}}(\boldsymbol{p}_2,t,t_f)\mathrm{i}G_{\text{ave,ave}}(\boldsymbol{p}_3,t,t_f)\mathrm{i}G_{\text{ave,ave}}(\boldsymbol{p}_4,t,t_f)+ \\ &+\text{permutations} =\end{aligned}$$

$$= -4\mathrm{i}\lambda \int_{t_0}^{t_f} \mathrm{d}t\, \frac{1}{E} \sin\big(E(t-t_f)\big) \underbrace{\Theta(t_f - t)}_{=1} \left(-\frac{\mathrm{i}}{2E} \cos\big(E(t-t_f)\big) \coth\left(\frac{E\beta}{2}\right)\right)^3 =$$

$$= -\frac{\lambda}{8E^5} \coth^3\left(\frac{E\beta}{2}\right) \big(1 - \cos^4(E\Delta t)\big) \tag{5.54}$$

where $\Delta t = t_f - t_0$.

$$(5.52) = -\mathrm{i}\frac{\lambda}{4} \int_{t_0}^{t_f} \mathrm{d}t\, \mathrm{i}G_{\mathrm{ave,ave}}(\boldsymbol{p}_1, t, t_f) \mathrm{i}G_{\mathrm{dif,ave}}(\boldsymbol{p}_2, t, t_f) \mathrm{i}G_{\mathrm{dif,ave}}(\boldsymbol{p}_3, t, t_f) \mathrm{i}G_{\mathrm{dif,ave}}(\boldsymbol{p}_4, t, t_f) +$$
$$+ \text{permutations} =$$
$$= -\mathrm{i}\lambda \int_{t_0}^{t_f} \mathrm{d}t\, \frac{-\mathrm{i}}{2E} \cos\big(E(t-t_f)\big) \coth\left(\frac{E\beta}{2}\right) \left(\frac{1}{E} \sin\big(E(t-t_f)\big) \underbrace{\Theta(t_f - t)}_{=1} \right)^3 =$$
$$= \frac{\lambda}{8E^5} \coth\left(\frac{E\beta}{2}\right) \sin^4(E\Delta t) \tag{5.55}$$

$$(5.53) = -\lambda \int_0^{\beta} \mathrm{d}\tau\, \mathrm{i}G_{\mathrm{ave,E}}(\boldsymbol{p}_1, t_f, \tau) \mathrm{i}G_{\mathrm{ave,E}}(\boldsymbol{p}_2, t_f, \tau) \mathrm{i}G_{\mathrm{ave,E}}(\boldsymbol{p}_3, t_f, \tau) \mathrm{i}G_{\mathrm{ave,E}}(\boldsymbol{p}_4, t_f, \tau) =$$
$$= -\lambda \int_0^{\beta} \mathrm{d}\tau \left(\frac{-\mathrm{i}}{2E} \cos\left(E\left(\Delta t + \mathrm{i}\tau - \mathrm{i}\frac{\beta}{2} \right) \right) \mathrm{csch}\left(\frac{E\beta}{2}\right) \right)^4 =$$
$$= -\frac{\lambda}{256E^5} \mathrm{csch}^4\left(\frac{E\beta}{2}\right) (6\beta E + 8\cos(2\Delta t)\sinh(\beta E) + \cos(4\Delta t)\sinh(2\beta E)) \tag{5.56}$$

Therefore, the total answer is

$$(5.51) + (5.52) + (5.53) = -\frac{\lambda}{256E^5} \mathrm{csch}^4\left(\frac{\beta E}{2}\right) (6\beta E + 8\sinh(\beta E) + \sinh(2\beta E)) \tag{5.57}$$

This end result is completely independent of time and fully agrees with an imaginary-time formalism calculation as it should. However, that time independence was once more only there due to the cross-terms. What is more, it is more accurate to say the real-time terms canceled the time dependence of the cross terms as the final answer comes purely from the cross terms. This is not recoverable from a modification of the quadratic components or

the 2×2 propagator matrix[6]. Further, it is now completely transparent that in no way the non-Gaussianities of the initial density matrix are damped or disappear at early times, in fact they are completely independent of time.

The only reasonable conclusion is that finite temperature quantum field theories are not free in the far past and that, if we wish to calculate higher point functions we must use the full 3×3 propagator matrix.

5.6 Conclusion

We conclude by contrasting this work with what is found in the pre-existing body of literature.

The first main difference with the most common approaches is that, so far, we have not relied too heavily on transforming to Fourier space in time. This difference is mostly cosmetic but there are reasons behind the choice made in this chapter.

Firstly, a priori, all our time variables exist in a compact time interval, either $[t_0, t_f]$ or $[0, \beta]$, therefore, naively, we cannot just Fouier transform.

However, we might wish to take a Fourier series instead. This is complicated by the fact none of our functions is periodic in these intervals individually. If we performed a Fourier series we would either ruin the boundary conditions for the value of the function or for its first derivative, we cannot keep both arbitrary.

Finally, one might want to leverage the fact the boundary conditions are joined in a loop as if the time variable was merely following a contour in the complex plane. This is perfectly legitimate in non-relativistic theories, which have first order equations of motion. However, for relativistic theories we run into a problem with matching the first derivatives. The issue is that, in order for this picture to work we would need to impose continuity of the first derivatives along the contour, which would actually mean imposing:

$$\dot{q}_+(t_f) = -\dot{q}_-(t_f) \tag{5.58}$$

which does not cancel the boundary terms when integrating by parts.

These subtleties may be ameliorated if one takes the limits $t_0 \to -\infty$ and $t_f \to \infty$, but we do not wish to do at this stage to make sure we have not been sloppy with these limits. This is ultimately why we avoid going to temporal Fourier space and mostly do not speak in terms of the time contour.

[6] See note in introduction for recent developments on the validity of this claim.

On a related point, the average-difference basis is not the only basis which can provide simplifications. Namely, there is the retarded-advanced basis [168, 195] which takes advantage of the Kubo-Martin-Schwinger (KMS) relation:

$$G_{+-}(t_1 + \mathrm{i}\beta, t_2) = G_{-+}(t_1, t_2) \tag{5.59}$$

However, this relates functions at different points in time, therefore it can only be easily used in Fourier space. For the reasons stated above we have avoided Fourier space and therefore not used the retarded-advanced basis. It is still important to note that there is even further structure in the propagators used in this chapter.

The most important difference with the pre-existing literature is the treatment of the cross terms between the real and imaginary segments. In the vast majority of the literature they are simply disregarded [159, 160, 162, 163, 165–168, 170–173, 177]. There are several arguments that are used to justify not taking them into consideration, but, in essence, they boil down to taking the limit $t_0 \to -\infty$ and either just assuming the interactions decay at very early times [160, 162] or changing the dynamics explicitly to forcibly turn off the interactions in the far past [186, 187].

Up to an extent this is perfectly legitimate. After all we can use whatever Hamiltonian we wish and whatever initial conditions we wish. There is no mathematical or physical inconsistency with choosing the initial density matrix to be $\rho = \exp(-\beta H_0)$, where H_0 is the quadratic part of the Hamiltonian, or adding an exponential decay to the interaction Hamiltonian. The real question is whether or not this is accurately capturing thermal physics.

If one used the ad-hoc $\rho = \exp(-\beta H_0)$ the issue is that, in contrast with the full Gibbs state, it is not time independent, the free Hamiltonian does not commute with the interaction Hamiltonian. Therefore we would have to trust this state is in some sense 'close enough' to the true finite temperature state so that the difference in observables calculated with either state would small or decaying with time. In sections 5.4 and 5.5 we have explicitly compared these two methods and reached the conclusion no such mechanism appears to exist[7].

If one changed the Hamiltonian to turn off the interactions there are two ways in which we could test its accuracy at describing thermal physics. The first is by comparing with experimental results. The second is to take the limit in which this damping is removed, which is what is usually described as desired [186, 187]. The issue with this last method is that the two limits may not commute. We may get different answers if we remove the damping before or after taking the $t_0 \to -\infty$. The calculations in sections 5.4 and 5.5 indeed demonstrate this will be the case[8].

[7] See note in introduction for recent developments on the validity of this claim.

[8] See note in introduction for recent developments on the validity of this claim.

There have also been some works in the past that tried to take the effect of the interactions into account [188, 189, 194, 198]. Most notably, in the non-relativistic community these effects have been widely studied and it is even a matter of textbooks and reviews [197, 199–201]. In this case it has even been argued that the 3×3 propagator matrix is equivalent to including an explicit coupling term to an external bath [201]. Nevertheless, the lessons from this case cannot be straightforwardly imported to relativistic theories. The main objection being that the propagator equations are first order in time which means time contour arguments are much more straightforward. The solutions are just distinct and there is no a priori reason that the arguments and proofs that work in that case can be extended to the relativistic case.

Another relatively known approach is that in [188, 189] which attempts to give a prescription for how to modify the 2×2 propagators into giving the full answer. However, the arguments do not quite hold up to scrutiny as they do not correctly take into account the presence of internal Euclidean vertices. Indeed as the calculations in section 5.5 demonstrate, no such reasoning can be true[9].

Finally, in [194] the role of the interactions is correctly taken into account and t_0 is held fixed until the very end by using a 2PI formalism. Unfortunately, none of the relativistic works that cite them correctly take interactions into account instead using the incorrect 2×2 propagator matrix. In [198] these effects are also taken into account but the technical points are mixed in with the disorder averaging, which complicates the interpretation.

All in all, despite the existence of some works which do take these effects into account misconceptions regarding the role of these interactions are overwhelming prevalent in the literature. The most popular textbooks and reviews, even recent ones, do not take these effects into account. The author hopes this work can demonstrate in a simple manner the importance of the cross-terms and clear the confusion in the field.

[9] See note in introduction for recent developments on the validity of this claim.

Chapter 6

Conclusion

After all of this work, we have understood quite a lot but there are still many questions left unanswered.

We began with a review of renormalisation. In short, the idea is that we introduce a regulator to bring 'infinity' to 'big'; then, as a consistency condition, we construct the RG flow (*e.g.* by integrating out UV modes); finally we use this flow to tune the regulator to improve perturbation theory and bring 'big' to 'small'. Counterterms arise as more generic way to encode this procedure which allows us to figure out some structure, for example, which couplings do we need to change and which can be left out. Finally, we described effective field theories as a means to justify and construct a hierarchy among irrelevant couplings, so that we can consider them one by one.

The main idea we left out was that of symmetry. For example, despite the initial intention to keep everything in Lorentzian signature, we quickly caved in and Wick rotated to Euclidean signature. Jumping to chapter 5 we see that for finite temperature it is fine to never Wick rotate and just cut off spatial momenta. However, the initial state explicitly breaks Lorentz invariance so there was no reason to even try to preserve this symmetry. It would nevertheless be interesting to explore the consequences of insisting on never Wick rotating.

We followed this by a study of AdS/CFT, first trying to tease out how could we view holographic renormalisation in a Wilsonian fashion. Using methods from the exact RG framework, which allow us to use smooth cut-offs, we managed to construct a consistent local RG scheme. We then used that to rule out quantum RG.

In this chapter the result is null so the question left open is obvious: can we write something like chapter 2 but regarding holography? Further, it would be interesting to revisit local RG in greater generality and see if it could be of use.

We then proceeded with a test of the correspondence. The idea is simple even if the application was messy. If the entropy calculations are independent from the gauge theory

coupling (indeed that is why they were possible) then they should also be independent from stringy corrections in the gravitational side. We confirmed this and even confirmed that away from the strict supersymmetric limit the corrections are according to the weak gravity conjecture.

This result speaks for itself and is quite conclusive, but there are hints of a greater structure. Would this work for a more generic black hole? What does this mean for the corrected solution?

Finally we delved into the world of finite temperature quantum field theory. It may seem like an odd detour but it was absolutely necessary. The whole objective was to understand secular divergences in black hole spacetimes. But, in order to do that, we needed a very solid understanding of finite temperature QFT and specifically how to set up initial conditions at a finite time in the past. It was imperative to clear up the abundant misconceptions in the literature regarding interactions in the initial state. The answer was clear, there is no sensible way in which we can neglect interactions in the initial state, even if it is set up in the infinite past. For some calculations those contributions are even the only non-zero ones.

From here on the road is clear. We just need to continue to apply this formalism to increasingly complicated spacetimes. Starting with the flat spacetime limit, moving to Rindler, and finally tackling Schwarzschild. Perhaps this road will even serve us to contribute towards cosmological applications. Especially less studied questions such as how to deal with the time dependence of the couplings in the effective field theory of inflation.

In the course of these 4 years and 100 pages we deepened our understanding of RG; we tested quantum RG and developed local RG; we computed stringy corrections and matched the CFT prediction; and we cleared up the fate of interactions in the far past at finite temperature. In some sense, it almost feels like there are more questions that were left open than those that were answered. But that is the nature of learning and of research. It would be somewhat boring if this was all there was to it wouldn't it?

References

[1] J. a. F. Melo and J. E. Santos, *Developing local RG: quantum RG and BFSS*, JHEP **05**, 063 (2020), doi:10.1007/JHEP05(2020)063, 1910.09559.

[2] J. a. F. Melo and J. E. Santos, *Stringy corrections to the entropy of electrically charged supersymmetric black holes with* $AdS_5 \times S^5$ *asymptotics*, Phys. Rev. D **103**(6), 066008 (2021), doi:10.1103/PhysRevD.103.066008, 2007.06582.

[3] M. F. Paulos, *Higher derivative terms including the Ramond-Ramond five-form*, JHEP **10**, 047 (2008), doi:10.1088/1126-6708/2008/10/047, 0804.0763.

[4] J. F. Donoghue, M. M. Ivanov and A. Shkerin, *EPFL Lectures on General Relativity as a Quantum Field Theory* (2017), 1702.00319.

[5] R. M. Wald, *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*, Chicago Lectures in Physics. University of Chicago Press, Chicago, IL, ISBN 978-0-226-87027-4 (1995).

[6] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, doi:10.1017/CBO9780511622632 (1982).

[7] A. A. Starobinsky, *STOCHASTIC DE SITTER (INFLATIONARY) STAGE IN THE EARLY UNIVERSE*, Lect. Notes Phys. **246**, 107 (1986), doi:10.1007/3-540-16452-9_6.

[8] C. P. Burgess, L. Leblond, R. Holman and S. Shandera, *Super-Hubble de Sitter Fluctuations and the Dynamical RG*, JCAP **03**, 033 (2010), doi:10.1088/1475-7516/2010/03/033, 0912.1608.

[9] C. P. Burgess, R. Holman, L. Leblond and S. Shandera, *Breakdown of Semiclassical Methods in de Sitter Space*, JCAP **10**, 017 (2010), doi:10.1088/1475-7516/2010/10/017, 1005.3551.

[10] E. T. Akhmedov, F. K. Popov and V. M. Slepukhin, *Infrared dynamics of the massive $\phi 4$ theory on de Sitter space*, Phys. Rev. D **88**, 024021 (2013), doi:10.1103/PhysRevD.88.024021, 1303.1068.

[11] C. P. Burgess, R. Holman, G. Tasinato and M. Williams, *EFT Beyond the Horizon: Stochastic Inflation and How Primordial Quantum Fluctuations Go Classical*, JHEP **03**, 090 (2015), doi:10.1007/JHEP03(2015)090, 1408.5002.

[12] C. P. Burgess, R. Holman and G. Tasinato, *Open EFTs, IR effects \& late-time resummations: systematic corrections in stochastic inflation*, JHEP **01**, 153 (2016), doi:10.1007/JHEP01(2016)153, 1512.00169.

[13] E. T. Akhmedov, U. Moschella, K. E. Pavlenko and F. K. Popov, *Infrared dynamics of massive scalars from the complementary series in de Sitter space*, Phys. Rev. D **96**(2), 025002 (2017), doi:10.1103/PhysRevD.96.025002, 1701.07226.

[14] E. T. Akhmedov, U. Moschella and F. K. Popov, *Characters of different secular effects in various patches of de Sitter space*, Phys. Rev. D **99**(8), 086009 (2019), doi:10.1103/PhysRevD.99.086009, 1901.07293.

[15] M. Baumgart and R. Sundrum, *De Sitter Diagrammar and the Resummation of Time*, JHEP **07**, 119 (2020), doi:10.1007/JHEP07(2020)119, 1912.09502.

[16] V. Gorbenko and L. Senatore, $\lambda\phi^4$ *in dS* (2019), 1911.00022.

[17] T. Cohen and D. Green, *Soft de Sitter Effective Theory*, JHEP **12**, 041 (2020), doi:10.1007/JHEP12(2020)041, 2007.03693.

[18] D. Green and A. Premkumar, *Dynamical RG and Critical Phenomena in de Sitter Space*, JHEP **04**, 064 (2020), doi:10.1007/JHEP04(2020)064, 2001.05974.

[19] A. A. Radovskaya and A. G. Semenov, *Semiclassical approximation meets Keldysh–Schwinger diagrammatic technique: scalar* φ^4, Eur. Phys. J. C **81**(8), 704 (2021), doi:10.1140/epjc/s10052-021-09382-4, 2003.06395.

[20] K. V. Bazarov, *Notes on peculiarities of quantum fields in space-times with horizons* (2021), 2112.02188.

[21] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *The entropy of Hawking radiation*, Rev. Mod. Phys. **93**(3), 035002 (2021), doi:10.1103/RevModPhys.93.035002, 2006.06872.

[22] T. D. Brennan, F. Carta and C. Vafa, *The String Landscape, the Swampland, and the Missing Corner*, PoS **TASI2017**, 015 (2017), doi:10.22323/1.305.0015, 1711.00864.

[23] E. Palti, *The Swampland: Introduction and Review*, Fortsch. Phys. **67**(6), 1900037 (2019), doi:10.1002/prop.201900037, 1903.06239.

[24] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, Phys. Rept. **989**, 1 (2022), doi:10.1016/j.physrep.2022.09.002, 2102.01111.

[25] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int.J.Theor.Phys. **38**, 1113 (1999), doi:10.1023/A:1026654312961, hep-th/9711200.

[26] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, Phys. Lett. **B428**, 105 (1998), doi:10.1016/S0370-2693(98)00377-3, hep-th/9802109.

[27] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.

[28] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. **323**, 183 (2000), doi:10.1016/S0370-1573(99)00083-6, hep-th/9905111.

[29] M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications*, Cambridge University Press, Cambridge, ISBN 978-1-107-01034-5, 978-1-316-23594-2 (2015).

[30] J. F. Melo, *Introduction to Renormalisation* (2019), 1909.11099.

[31] J. a. F. Melo, *The Propagator Matrix Reloaded* (2021), 2112.09119.

[32] D. Skinner, *Quantum Field Theory II*, http://www.damtp.cam.ac.uk/user/dbs26/AQFT.html (2017).

[33] Y. BenTov, *Schwinger-Keldysh path integral for the quantum harmonic oscillator* (2021), 2102.05029.

[34] D. Tong, *Statistical Field Theory*, http://www.damtp.cam.ac.uk/user/tong/sft.html (2017).

[35] C. E. Thomas, *The Standard Model*, http://www.damtp.cam.ac.uk/user/cet34/teaching/SM/ (2019).

[36] T. Weigand, *Quantum Field Theory I + II*, http://www.physics.umd.edu/grt/taj/624b/WeigandQFT.pdf (2017).

[37] C. P. Burgess, *Introduction to Effective Field Theory*, Cambridge University Press, ISBN 978-1-139-04804-0, 978-0-521-19547-8, doi:10.1017/9781139048040 (2020).

[38] A. A. Petrov and A. E. Blechman, *Effective Field Theories*, WSP, ISBN 978-981-4434-92-8, 978-981-4434-94-2, doi:10.1142/8619 (2016).

[39] B. Delamotte, *A Hint of renormalization*, Am. J. Phys. **72**, 170 (2004), doi:10.1119/1.1624112, hep-th/0212049.

[40] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, doi:10.1017/9781139540940 (2013).

[41] S.-S. Lee, *Quantum renormalization group and holography*, Journal of High Energy Physics **2014**, 76 (2014), doi:10.1007/JHEP01(2014)076, 1305.3908.

[42] V. Balasubramanian and P. Kraus, *Spacetime and the Holographic Renormalization Group*, Phys. Rev. Lett. **83**(18), 3605 (1999), doi:10.1103/PhysRevLett.83.3605, hep-th/9903190.

[43] S. de Haro, S. N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, Commun. Math. Phys. **217**, 595 (2001), doi:10.1007/s002200100381, hep-th/0002230.

[44] K. Skenderis, *Lecture Notes on Holographic Renormalization*, arXiv e-prints hep-th/0209067 (2002), hep-th/0209067.

[45] I. Kanitscheider, K. Skenderis and M. Taylor, *Precision holography for non-conformal branes*, Journal of High Energy Physics **2008**(9), 094 (2008), doi:10.1088/1126-6708/2008/09/094, 0807.3324.

[46] I. Papadimitriou, *Lectures on Holographic Renormalization*, Springer Proc. Phys. **176**, 131 (2016), doi:10.1007/978-3-319-31352-8_4.

[47] E. T. Akhmedov, *A remark on the AdS/CFT correspondence and the renormalization group flow*, Physics Letters B **442**(1-4), 152 (1998), doi:10.1016/S0370-2693(98)01270-2, hep-th/9806217.

[48] K. G. Wilson and J. Kogut, *The renormalization group and the ε expansion*, Physics Reports **12**(2), 75 (1974), doi:https://doi.org/10.1016/0370-1573(74)90023-4.

[49] E. Verlinde and H. Verlinde, *RG-flow, gravity and the cosmological constant*, Journal of High Energy Physics **2000**(5), 034 (2000), doi:10.1088/1126-6708/2000/05/034, hep-th/9912018.

[50] J. de Boer, E. Verlinde and H. Verlinde, *On the holographic renormalization group*, Journal of High Energy Physics **2000**(8), 003 (2000), doi:10.1088/1126-6708/2000/08/003, hep-th/9912012.

[51] J. de Boer, *The Holographic Renormalization Group*, Fortschritte der Physik **49**(4), 339 (2001), doi:10.1002/1521-3978(200105)49:4/6<339::AID-PROP339>3.0.CO;2-A, hep-th/0101026.

[52] I. Heemskerk and J. Polchinski, *Holographic and Wilsonian renormalization groups*, Journal of High Energy Physics **2011**, 31 (2011), doi:10.1007/JHEP06(2011)031, 1010.1264.

[53] T. Faulkner, H. Liu and M. Rangamani, *Integrating out geometry: holographic Wilsonian RG and the membrane paradigm*, Journal of High Energy Physics **2011**, 51 (2011), doi:10.1007/JHEP08(2011)051, 1010.4036.

[54] S. Grozdanov, *Wilsonian renormalisation and the exact cut-off scale from holographic duality*, Journal of High Energy Physics **2012**, 79 (2012), doi:10.1007/JHEP06(2012)079, 1112.3356.

[55] V. Balasubramanian, M. Guica and A. Lawrence, *Holographic interpretations of the renormalization group*, Journal of High Energy Physics **2013**, 115 (2013), doi:10.1007/JHEP01(2013)115, 1211.1729.

[56] J. M. Lizana, T. R. Morris and M. Pérez-Victoria, *Holographic renormalisation group flows and renormalisation from a Wilsonian perspective*, Journal of High Energy Physics **2016**(3), 198 (2016), doi:10.1007/JHEP03(2016)198, 1511.04432.

[57] F. Ardalan, *Cutoff function in holographic RG flow*, Phys. Rev. D **99**(10), 106016 (2019), doi:10.1103/PhysRevD.99.106016, 1902.09123.

[58] A. Bzowski and M. Schillo, *Dimensional regularization for holographic RG flows*, arXiv e-prints arXiv:1906.02234 (2019), 1906.02234.

[59] S.-S. Lee, *Holographic description of quantum field theory*, Nuclear Physics B **832**(3), 567 (2010), doi:10.1016/j.nuclphysb.2010.02.022, 0912.5223.

[60] S.-S. Lee, *Background independent holographic description: from matrix field theory to quantum gravity*, Journal of High Energy Physics **2012**, 160 (2012), doi:10.1007/JHEP10(2012)160, 1204.1780.

[61] I. Papadimitriou and K. Skenderis, *AdS / CFT correspondence and geometry*, IRMA Lect. Math. Theor. Phys. **8**, 73 (2005), doi:10.4171/013-1/4, hep-th/0404176.

[62] I. Papadimitriou, *Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT*, JHEP **05**, 075 (2007), doi:10.1088/1126-6708/2007/05/075, hep-th/0703152.

[63] I. Papadimitriou, *Holographic renormalization as a canonical transformation*, JHEP **11**, 014 (2010), doi:10.1007/JHEP11(2010)014, 1007.4592.

[64] I. Papadimitriou, *Holographic Renormalization of general dilaton-axion gravity*, JHEP **08**, 119 (2011), doi:10.1007/JHEP08(2011)119, 1106.4826.

[65] N. Behr, S. Kuperstein and A. Mukhopadhyay, *Holography as a highly efficient renormalization group flow. I. Rephrasing gravity*, Phys. Rev. D **94**(2), 026001 (2016), doi:10.1103/PhysRevD.94.026001, 1502.06619.

[66] N. Behr and A. Mukhopadhyay, *Holography as a highly efficient renormalization group flow. II. An explicit construction*, Phys. Rev. D **94**(2), 026002 (2016), doi:10.1103/PhysRevD.94.026002, 1512.09055.

[67] G. Mandal and P. Nayak, *Revisiting AdS/CFT at a finite radial cut-off*, JHEP **12**, 125 (2016), doi:10.1007/JHEP12(2016)125, 1608.00411.

[68] A. Mukhopadhyay, *Understanding the holographic principle via RG flow*, International Journal of Modern Physics A **31**(34), 1630059-203 (2016), doi:10.1142/S0217751X16300593, 1612.00141.

[69] B. Sathiapalan and H. Sonoda, *A holographic form for Wilson's RG*, Nuclear Physics B **924**, 603 (2017), doi:10.1016/j.nuclphysb.2017.09.018, 1706.03371.

[70] B. Sathiapalan and H. Sonoda, *Holographic Wilson's RG*, arXiv e-prints arXiv:1902.02486 (2019), 1902.02486.

[71] Y. Nakayama, *Local renormalization group functions from quantum renormalization group and holographic bulk locality*, Journal of High Energy Physics **2015**, 92 (2015), doi:10.1007/JHEP06(2015)092, 1502.07049.

[72] P. Lunts, S. Bhattacharjee, J. Miller, E. Schnetter, Y. B. Kim and S.-S. Lee, *Ab initio holography*, Journal of High Energy Physics **2015**, 107 (2015), doi:10.1007/JHEP08(2015)107, 1503.06474.

[73] V. Shyam, *General covariance from the quantum renormalization group*, Phys. Rev. D **95**(6), 066003 (2017), doi:10.1103/PhysRevD.95.066003, 1611.05315.

[74] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. D **55**(8), 5112 (1997), doi:10.1103/PhysRevD.55.5112, hep-th/9610043.

[75] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, Phys. Rev. D **58**(4), 046004 (1998), doi:10.1103/PhysRevD.58.046004, hep-th/9802042.

[76] Y. Sekino and T. Yoneya, *Generalized AdS-CFT correspondence for Matrix theory in the large-/N limit*, Nuclear Physics B **570**(1), 174 (2000), doi:10.1016/S0550-3213(99)00793-2, hep-th/9907029.

[77] T. Ortiz and H. Samtleben, *SO(9) supergravity in two dimensions*, Journal of High Energy Physics **2013**, 183 (2013), doi:10.1007/JHEP01(2013)183, 1210.4266.

[78] A. Anabalón, T. Ortiz and H. Samtleben, *Rotating D0-branes and consistent truncations of supergravity*, Physics Letters B **727**(4-5), 516 (2013), doi:10.1016/j.physletb.2013.10.049, 1310.1321.

[79] T. Ortiz, H. Samtleben and D. Tsimpis, *Matrix model holography*, Journal of High Energy Physics **2014**, 96 (2014), doi:10.1007/JHEP12(2014)096, 1410.0487.

[80] M. Hanada, J. Nishimura, Y. Sekino and T. Yoneya, *Direct test of the gauge-gravity correspondence for Matrix theory correlation functions*, Journal of High Energy Physics **2011**, 20 (2011), doi:10.1007/JHEP12(2011)020, 1108.5153.

[81] B. Ydri, *Review of M(atrix)-Theory, Type IIB Matrix Model and Matrix String Theory*, arXiv e-prints arXiv:1708.00734 (2017), 1708.00734.

[82] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, USA, ISBN 9780201503975, 0201503972 (1995).

[83] Y. Igarashi, K. Itoh and H. Sonoda, *Realization of Symmetry in the ERG Approach to Quantum Field Theory*, Progress of Theoretical Physics Supplement **181**, 1 (2009), doi:10.1143/PTPS.181.1, 0909.0327.

[84] T. R. Morris, *Elements of the Continuous Renormalization Group*, Progress of Theoretical Physics Supplement **131**, 395 (1998), doi:10.1143/PTPS.131.395, hep-th/9802039.

[85] C. Bagnuls and C. Bervillier, *Exact renormalization group equations: an introductory review*, "Physics Reports" **348**(1-2), 91 (2001), doi:10.1016/S0370-1573(00)00137-X, hep-th/0002034.

[86] J. Berges, N. Tetradis and C. Wetterich, *Non-perturbative renormalization flow in quantum field theory and statistical physics*, "Physics Reports" **363**(4-6), 223 (2002), doi:10.1016/S0370-1573(01)00098-9, hep-ph/0005122.

[87] J. Polonyi, *Lectures on the functional renormalization group method*, Central European Journal of Physics **1**, 1 (2003), doi:10.2478/BF02475552, hep-th/0110026.

[88] O. J. Rosten, *Fundamentals of the exact renormalization group*, "Physics Reports" **511**(4), 177 (2012), doi:10.1016/j.physrep.2011.12.003, 1003.1366.

[89] I. Taylor, Washington and M. Van Raamsdonk, *Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds*, Journal of High Energy Physics **1999**(4), 013 (1999), doi:10.1088/1126-6708/1999/04/013, hep-th/9812239.

[90] A. Jevicki and T. Yoneya, *Space-time uncertainty principle and conformal symmetry in D-particle dynamics*, Nuclear Physics B **535**(1), 335 (1998), doi:10.1016/S0550-3213(98)00578-1, hep-th/9805069.

[91] A. Jevicki, Y. Kazama and T. Yoneya, *Generalized conformal symmetry in D-brane matrix models*, Phys. Rev. D **59**(6), 066001 (1999), doi:10.1103/PhysRevD.59.066001, hep-th/9810146.

[92] K. Hayasaka and R. Nakayama, *Conformal invariance of the D-particle effective action*, Physics Letters B **463**(2-4), 181 (1999), doi:10.1016/S0370-2693(99)00990-9, hep-th/9904083.

[93] H. Hata and S. Moriyama, *Conformal symmetry and a new gauge in the matrix model*, Phys. Rev. D **60**(12), 126006 (1999), doi:10.1103/PhysRevD.60.126006, hep-th/9904042.

[94] M. Taylor, *Generalized conformal structure, dilaton gravity and SYK*, Journal of High Energy Physics **2018**(1), 10 (2018), doi:10.1007/JHEP01(2018)010, 1706.07812.

[95] D. Christodoulou, *Reversible and irreversible transforations in black hole physics*, Phys. Rev. Lett. **25**, 1596 (1970), doi:10.1103/PhysRevLett.25.1596.

[96] S. Hawking, *Gravitational radiation from colliding black holes*, Phys. Rev. Lett. **26**, 1344 (1971), doi:10.1103/PhysRevLett.26.1344.

[97] B. Carter, *Rigidity of a black hole*, Nature Physical Science **238**(83), 71 (1972), doi:10.1038/physci238071b0.

[98] J. Bekenstein, *Black holes and the second law*, Lett. Nuovo Cim. **4**, 737 (1972), doi:10.1007/BF02757029.

[99] D. Christodoulou and R. Ruffini, *Reversible transformations of a charged black hole*, Phys. Rev. D **4**, 3552 (1971), doi:10.1103/PhysRevD.4.3552.

[100] J. D. Bekenstein, *Black holes and entropy*, Phys. Rev. D **7**, 2333 (1973), doi:10.1103/PhysRevD.7.2333.

[101] J. M. Bardeen, B. Carter and S. Hawking, *The Four laws of black hole mechanics*, Commun. Math. Phys. **31**, 161 (1973), doi:10.1007/BF01645742.

[102] S. Hawking, *Black hole explosions*, Nature **248**, 30 (1974), doi:10.1038/248030a0.

[103] S. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43**, 199 (1975), doi:10.1007/BF02345020, [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[104] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. B **379**, 99 (1996), doi:10.1016/0370-2693(96)00345-0, hep-th/9601029.

[105] J. Breckenridge, D. Lowe, R. C. Myers, A. Peet, A. Strominger and C. Vafa, *Macroscopic and microscopic entropy of near extremal spinning black holes*, Phys. Lett. B **381**, 423 (1996), doi:10.1016/0370-2693(96)00553-9, hep-th/9603078.

[106] C. G. Callan and J. M. Maldacena, *D-brane approach to black hole quantum mechanics*, Nucl. Phys. B **472**, 591 (1996), doi:10.1016/0550-3213(96)00225-8, hep-th/9602043.

[107] G. T. Horowitz and A. Strominger, *Counting states of near extremal black holes*, Phys. Rev. Lett. **77**, 2368 (1996), doi:10.1103/PhysRevLett.77.2368, hep-th/9602051.

[108] M. Cyrier, M. Guica, D. Mateos and A. Strominger, *Microscopic entropy of the black ring*, Phys. Rev. Lett. **94**, 191601 (2005), doi:10.1103/PhysRevLett.94.191601, hep-th/0411187.

[109] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, Commun. Math. Phys. **275**, 209 (2007), doi:10.1007/s00220-007-0258-7, hep-th/0510251.

[110] S. M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS_5*, JHEP **07**, 106 (2017), doi:10.1007/JHEP07(2017)106, 1705.05383.

[111] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, JHEP **10**, 062 (2019), doi:10.1007/JHEP10(2019)062, 1810.11442.

[112] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT* (2018), 1810.12067.

[113] F. Benini and P. Milan, *Black holes in 4d $\mathcal{N} = 4$ Super-Yang-Mills*, Phys. Rev. X **10**(2), 021037 (2020), doi:10.1103/PhysRevX.10.021037, 1812.09613.

[114] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, Phys. Rev. D **100**(2), 026008 (2019), doi:10.1103/PhysRevD.100.026008, 1901.08091.

[115] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS_5 blackholes*, JHEP **06**, 134 (2019), doi:10.1007/JHEP06(2019)134, 1902.06619.

[116] A. Zaffaroni, *Lectures on AdS Black Holes, Holography and Localization*, In *CERN Winter School on Strings and Fields 2017 and ICTP School on Supersymmetric Localization, Holography and Related Topics* (2019), 1902.07176.

[117] J. Kim, S. Kim and J. Song, *A 4d $N = 1$ Cardy Formula* (2019), 1904.03455.

[118] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The asymptotic growth of states of the 4d $\mathcal{N} = 1$ superconformal index*, JHEP **08**, 120 (2019), doi:10.1007/JHEP08(2019)120, 1904.05865.

[119] A. González Lezcano and L. A. Pando Zayas, *Microstate counting via Bethe Ansätze in the 4d $\mathcal{N}$ = 1 superconformal index*, JHEP **03**, 088 (2020), doi: 10.1007/JHEP03(2020)088, 1907.12841.

[120] A. Lanir, A. Nedelin and O. Sela, *Black hole entropy function for toric theories via Bethe Ansatz*, JHEP **04**, 091 (2020), doi:10.1007/JHEP04(2020)091, 1908.01737.

[121] A. Cabo-Bizet and S. Murthy, *Supersymmetric phases of 4d N=4 SYM at large N* (2019), 1909.09597.

[122] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The large-N limit of the 4d $\mathcal{N} = 1$ superconformal index* (2020), 2005.10654.

[123] S. Murthy, *The growth of the $\frac{1}{16}$-BPS index in 4d $\mathcal{N} = 4$ SYM* (2020), 2005.10843.

[124] P. Agarwal, S. Choi, J. Kim, S. Kim and J. Nahmgoong, *AdS black holes and finite N indices* (2020), 2005.11240.

[125] F. Benini, E. Colombo, S. Soltani, A. Zaffaroni and Z. Zhang, *Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes* (2020), 2005.12308.

[126] A. Gadde, *Lectures on the Superconformal Index* (2020), 2006.13630.

[127] J. B. Gutowski and H. S. Reall, *Supersymmetric AdS(5) black holes*, JHEP **02**, 006 (2004), doi:10.1088/1126-6708/2004/02/006, hep-th/0401042.

[128] J. B. Gutowski and H. S. Reall, *General supersymmetric AdS(5) black holes*, JHEP **04**, 048 (2004), doi:10.1088/1126-6708/2004/04/048, hep-th/0401129.

[129] H. K. Kunduri, J. Lucietti and H. S. Reall, *Supersymmetric multi-charge AdS(5) black holes*, JHEP **04**, 036 (2006), doi:10.1088/1126-6708/2006/04/036, hep-th/0601156.

[130] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, JHEP **11**, 127 (2011), doi:10.1007/JHEP11(2011)127, hep-th/0304094.

[131] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing, and Mock Modular Forms* (2012), 1208.4074.

[132] M. B. Green and C. Stahn, *D3-branes on the Coulomb branch and instantons*, JHEP **09**, 052 (2003), doi:10.1088/1126-6708/2003/09/052, hep-th/0308061.

[133] M. Cvetic, H. Lu and C. Pope, *Charged Kerr-de Sitter black holes in five dimensions*, Phys. Lett. B **598**, 273 (2004), doi:10.1016/j.physletb.2004.08.011, hep-th/0406196.

[134] J. York, James W., *Role of conformal three geometry in the dynamics of gravitation*, Phys. Rev. Lett. **28**, 1082 (1972), doi:10.1103/PhysRevLett.28.1082.

[135] G. Gibbons and S. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, Phys. Rev. D **15**, 2752 (1977), doi:10.1103/PhysRevD.15.2752.

[136] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, Commun. Math. Phys. **208**, 413 (1999), doi:10.1007/s002200050764, hep-th/9902121.

[137] G. J. Loges, T. Noumi and G. Shiu, *Duality and Supersymmetry Constraints on the Weak Gravity Conjecture* (2020), 2006.06696.

[138] J. Markeviciute and J. E. Santos, *Hairy black holes in $AdS_5 \times S^5$*, JHEP **06**, 096 (2016), doi:10.1007/JHEP06(2016)096, 1602.03893.

[139] J. Markeviciute, *Rotating Hairy Black Holes in $AdS_5 \times S^5$*, JHEP **03**, 110 (2019), doi:10.1007/JHEP03(2019)110, 1809.04084.

[140] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. D **60**, 064018 (1999), doi:10.1103/PhysRevD.60.064018, hep-th/9902170.

[141] M. Cvetic, M. Duff, P. Hoxha, J. T. Liu, H. Lu, J. Lu, R. Martinez-Acosta, C. Pope, H. Sati and T. A. Tran, *Embedding AdS black holes in ten-dimensions and eleven-dimensions*, Nucl. Phys. B **558**, 96 (1999), doi:10.1016/S0550-3213(99)00419-8, hep-th/9903214.

[142] M. Cvetic, H. Lu, C. Pope, A. Sadrzadeh and T. A. Tran, *Consistent SO(6) reduction of type IIB supergravity on S**5*, Nucl. Phys. B **586**, 275 (2000), doi:10.1016/S0550-3213(00)00372-2, hep-th/0003103.

[143] H. S. Reall and J. E. Santos, *Higher derivative corrections to Kerr black hole thermodynamics*, JHEP **04**, 021 (2019), doi:10.1007/JHEP04(2019)021, 1901.11535.

[144] S.-Q. Wu, *General Nonextremal Rotating Charged AdS Black Holes in Five-dimensional $U(1)^3$ Gauged Supergravity: A Simple Construction Method*, Phys. Lett. B **707**, 286 (2012), doi:10.1016/j.physletb.2011.12.031, 1108.4159.

[145] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, JHEP **06**, 060 (2007), doi:10.1088/1126-6708/2007/06/060, hep-th/0601001.

[146] Y. Kats, L. Motl and M. Padi, *Higher-order corrections to mass-charge relation of extremal black holes*, JHEP **12**, 068 (2007), doi:10.1088/1126-6708/2007/12/068, hep-th/0606100.

[147] Y. Hamada, T. Noumi and G. Shiu, *Weak Gravity Conjecture from Unitarity and Causality*, Phys. Rev. Lett. **123**(5), 051601 (2019), doi:10.1103/PhysRevLett.123.051601, 1810.03637.

[148] C. Cheung, J. Liu and G. N. Remmen, *Proof of the Weak Gravity Conjecture from Black Hole Entropy*, JHEP **10**, 004 (2018), doi:10.1007/JHEP10(2018)004, 1801.08546.

[149] G. J. Loges, T. Noumi and G. Shiu, *Thermodynamics of 4D Dilatonic Black Holes and the Weak Gravity Conjecture*, Phys. Rev. D **102**, 046010 (2020), doi:10.1103/PhysRevD.102.046010, 1909.01352.

[150] G. Goon and R. Penco, *Universal Relation between Corrections to Entropy and Extremality*, Phys. Rev. Lett. **124**(10), 101103 (2020), doi:10.1103/PhysRevLett.124.101103, 1909.05254.

[151] S. Cremonini, C. R. Jones, J. T. Liu and B. McPeak, *Higher-Derivative Corrections to Entropy and the Weak Gravity Conjecture in Anti-de Sitter Space* (2019), 1912.11161.

[152] M. David, J. Nian and L. A. Pando Zayas, *Gravitational Cardy Limit and AdS Black Hole Entropy* (2020), 2005.10251.

[153] F. Larsen, J. Nian and Y. Zeng, *AdS_5 black hole entropy near the BPS limit*, JHEP **06**, 001 (2020), doi:10.1007/JHEP06(2020)001, 1907.02505.

[154] N. Bobev, A. M. Charles, K. Hristov and V. Reys, *The Unreasonable Effectiveness of Higher-Derivative Supergravity in AdS_4 Holography* (2020), 2006.09390.

[155] S. Weinberg, *The Quantum Theory of Fields*, vol. 1, Cambridge University Press, doi:10.1017/CBO9781139644167 (1995).

[156] D. Tong, *Lectures on Quantum Field Theory*, http://www.damtp.cam.ac.uk/user/tong/qft.html (2007).

[157] J. S. Schwinger, *Brownian motion of a quantum oscillator*, J. Math. Phys. **2**, 407 (1961), doi:10.1063/1.1703727.

[158] L. V. Keldysh, *Diagram technique for nonequilibrium processes*, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964).

[159] K.-c. Chou, Z.-b. Su, B.-l. Hao and L. Yu, *Equilibrium and Nonequilibrium Formalisms Made Unified*, Phys. Rept. **118**, 1 (1985), doi:10.1016/0370-1573(85)90136-X.

[160] N. P. Landsman and C. G. van Weert, *Real and Imaginary Time Field Theory at Finite Temperature and Density*, Phys. Rept. **145**, 141 (1987), doi:10.1016/0370-1573(87)90121-9.

[161] P. Aurenche, *Some aspects of thermal field theories*, In *4th Workshop in High-energy Phenomenology (WHEPP 4)* (1996), hep-ph/9612432.

[162] A. K. Das, *Finite Temperature Field Theory*, World Scientific, New York, ISBN 978-981-02-2856-9, 978-981-4498-23-4 (1997).

[163] M. Le Bellac, *Thermal Field Theory Michel Le Bellac.*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, ISBN 9780511721700 (2000).

[164] U. Kraemmer and A. Rebhan, *Advances in perturbative thermal field theory*, Rept. Prog. Phys. **67**, 351 (2004), doi:10.1088/0034-4885/67/3/R05, hep-ph/0310337.

[165] S. Jeon, *Color Glass Condensate in Schwinger-Keldysh QCD*, Annals Phys. **340**, 119 (2014), doi:10.1016/j.aop.2013.09.019, 1308.0263.

[166] J. Berges, *Nonequilibrium Quantum Fields: From Cold Atoms to Cosmology* (2015), 1503.02907.

[167] F. Gelis, *Initial state and thermalization in the Color Glass Condensate framework*, Int. J. Mod. Phys. E **24**(10), 1530008 (2015), doi:10.1142/S0218301315300088, 1508.07974.

[168] F. M. Haehl, R. Loganayagam and M. Rangamani, *Schwinger-Keldysh formalism. Part I: BRST symmetries and superspace*, JHEP **06**, 069 (2017), doi:10.1007/JHEP06(2017)069, 1610.01940.

[169] X. Chen, Y. Wang and Z.-Z. Xianyu, *Schwinger-Keldysh Diagrammatics for Primordial Perturbations*, JCAP **12**, 006 (2017), doi:10.1088/1475-7516/2017/12/006, 1703.10166.

[170] J. Ghiglieri, A. Kurkela, M. Strickland and A. Vuorinen, *Perturbative Thermal QCD: Formalism and Applications*, Phys. Rept. **880**, 1 (2020), doi:10.1016/j.physrep.2020.07.004, 2002.10188.

[171] T. Lundberg and R. Pasechnik, *Thermal Field Theory in real-time formalism: concepts and applications for particle decays*, Eur. Phys. J. A **57**(2), 71 (2021), doi:10.1140/epja/s10050-020-00288-5, 2007.01224.

[172] A. J. Niemi and G. W. Semenoff, *Thermodynamic Calculations in Relativistic Finite Temperature Quantum Field Theories*, Nucl. Phys. B **230**, 181 (1984), doi:10.1016/0550-3213(84)90123-8.

[173] A. J. Niemi and G. W. Semenoff, *Finite Temperature Quantum Field Theory in Minkowski Space*, Annals Phys. **152**, 105 (1984), doi:10.1016/0003-4916(84)90082-4.

[174] M. Le Bellac and H. Mabilat, *Infrared and pinching singularities in out-of-equilibrium QCD plasmas*, Z. Phys. C **75**, 137 (1997), doi:10.1007/s002880050455.

[175] P. Jizba, *Hydrostatic pressure of the O(N) phi**4 theory in the large N limit*, Phys. Rev. D **69**, 085011 (2004), doi:10.1103/PhysRevD.69.085011, hep-th/9801197.

[176] J. I. Kapusta, *Finite-temperature field theory.*, Cambridge monographs on mathematical physics. Cambridge University Press, Cambridge, 2nd ed. / joseph i. kapusta, charles gale. edn., ISBN 9780521173223 (2006).

[177] C. P. Burgess, J. Hainge, G. Kaplanek and M. Rummel, *Failure of Perturbation Theory Near Horizons: the Rindler Example*, JHEP **10**, 122 (2018), doi:10.1007/JHEP10(2018)122, 1806.11415.

[178] P. Sen, D. Indumathi and D. Choudhury, *Infrared finiteness of a complete theory of charged scalars and fermions at finite temperature*, Eur. Phys. J. C **80**(10), 972 (2020), doi:10.1140/epjc/s10052-020-08498-3.

[179] E. T. Akhmedov, *Curved space equilibration vs. flat space thermalization (a short review)* (2021), doi:10.1142/S0217732321300202, 2105.05039.

[180] E. T. Akhmedov, H. Godazgar and F. K. Popov, *Hawking radiation and secularly growing loop corrections*, Phys. Rev. D **93**(2), 024029 (2016), doi:10.1103/PhysRevD.93.024029, 1508.07500.

[181] G. Kaplanek and C. P. Burgess, *Hot Accelerated Qubits: Decoherence, Thermalization, Secular Growth and Reliable Late-time Predictions*, JHEP **03**, 008 (2020), doi:10.1007/JHEP03(2020)008, 1912.12951.

[182] G. Kaplanek and C. P. Burgess, *Hot Cosmic Qubits: Late-Time de Sitter Evolution and Critical Slowing Down*, JHEP **02**, 053 (2020), doi:10.1007/JHEP02(2020)053, 1912.12955.

[183] E. T. Akhmedov, P. A. Anempodistov, K. V. Bazarov, D. V. Diakonov and U. Moschella, *Heating up an environment around black holes and inside de Sitter space*, Phys. Rev. D **103**(2), 025023 (2021), doi:10.1103/PhysRevD.103.025023, 2010.10877.

[184] G. Kaplanek and C. P. Burgess, *Qubits on the Horizon: Decoherence and Thermalization near Black Holes*, JHEP **01**, 098 (2021), doi:10.1007/JHEP01(2021)098, 2007.05984.

[185] J. F. Melo, *Secular Divergences in Rindler and Minskowski Spacetimes*, In Preparation.

[186] M. Le Bellac and H. Mabilat, *Real time Feynman rules at finite temperature*, Phys. Lett. B **381**, 262 (1996), doi:10.1016/0370-2693(96)00604-1.

[187] R. van Leeuwen and G. Stefanucci, *Equilibrium and nonequilibrium many-body perturbation theory: a unified framework based on the Martin-Schwinger hierarchy*, In *Journal of Physics Conference Series*, vol. 427 of *Journal of Physics Conference Series*, p. 012001, doi:10.1088/1742-6596/427/1/012001 (2013), 1303.7105.

[188] F. Gelis, *The Effect of the vertical part of the path on the real time Feynman rules in finite temperature field theory*, Z. Phys. C **70**, 321 (1996), doi:10.1007/s002880050109, hep-ph/9412347.

[189] F. Gelis, *A New approach for the vertical part of the contour in thermal field theories*, Phys. Lett. B **455**, 205 (1999), doi:10.1016/S0370-2693(99)00460-8, hep-ph/9901263.

[190] A. Niegawa, *Out-of-equilibrium relativistic quantum field theory: Perturbation theory and progress of phase transition* (1998), hep-th/9810043.

[191] S. Banerjee, K. Papadodimas, S. Raju, P. Samantray and P. Shrivastava, *A Bound on Thermal Relativistic Correlators at Large Spacelike Momenta*, SciPost Phys. **8**(4), 064 (2020), doi:10.21468/SciPostPhys.8.4.064, 1902.07203.

[192] A. Radovskaya and A. G. Semenov, *Comment on the paper SciPost Phys. Core 6, 019 (2023) by João F. Melo (arXiv:2112.09119)* (2023), 2304.08536.

[193] F. Gelis and R. Venugopalan, *Particle production in field theories coupled to strong external sources*, Nucl. Phys. A **776**, 135 (2006), doi:10.1016/j.nuclphysa.2006.07.020, hep-ph/0601209.

[194] M. Garny and M. M. Muller, *Kadanoff-Baym Equations with Non-Gaussian Initial Conditions: The Equilibrium Limit*, Phys. Rev. D **80**, 085011 (2009), doi:10.1103/PhysRevD.80.085011, 0904.3600.

[195] M. A. van Eijck, R. Kobes and C. G. van Weert, *Transformations of real time finite temperature Feynman rules*, Phys. Rev. D **50**, 4097 (1994), doi:10.1103/PhysRevD.50.4097, hep-ph/9406214.

[196] Y. Takahasi and H. Umezawa, *Thermo field dynamics*, Collect. Phenom. **2**, 55 (1975).

[197] A. Kamenev, *Many-body theory of non-equilibrium systems*, arXiv e-prints cond-mat/0412296 (2004), cond-mat/0412296.

[198] L. F. Cugliandolo, G. S. Lozano and N. Nessi, *Role of initial conditions in the dynamics of quantum glassy systems*, J. Stat. Mech. **1902**, 023301 (2019), doi: 10.1088/1742-5468/ab02ed, 1811.03987.

[199] R. van Leeuwen, N. Dahlen, G. Stefanucci, C.-O. Almbladh and U. von Barth, *Introduction to the Keldysh Formalism*, pp. 33–59, Springer Berlin Heidelberg, Berlin, Heidelberg, ISBN 978-3-540-35426-0, doi:10.1007/3-540-35426-3_3 (2006).

[200] R. van Leeuwen and G. Stefanucci, *Wick Theorem for General Initial States*, Phys. Rev. B **85**, 115119 (2012), doi:10.1103/PhysRevB.85.115119, 1102.4814.

[201] P. I. Arseev, *On the nonequilibrium diagram technique: derivation, some features, and applications*, Physics-Uspekhi **58**(12), 1159 (2015), doi:10.3367/ufne.0185.201512b.1271.